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Trends in 20th Century Celestial Mechanics

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Abstract

We review some of the major achievements of celestial mechanics in the twentieth century, and discuss some unsolved problems that are left to the twenty-first century. The four major research fields in celestial mechanics are treated: dynamical systems, three-body problems, solar system dynamics, and numerical methods. Over the past decades, celestial mechanics has become extremely specialized and categorized along many trajectories, from purely mathematical to quite practical points of view, as well as in what has involved digital computer innovation. These various ends of celestial mechanics sometimes exist in isolation, losing their connections with each other. The purpose of this manuscript is to find clues for rediscovering connections between each of these fields by listing some of major achievements in hoping to find new paths for celestial mechanics in the new century.

Key words: celestial mechanics, dynamical system, three-body problem, planetary dynamics, numerical method

Contents

1. Introduction

This manuscript represents our effort to summarize some of the major trends in celestial mechanics in the recently ended twentieth century. Not only celestial mechanics but the entire field of astronomy has evolved and drastically changed over the past one hundred years. To commemorate the beginning of the new century, we venture this review of the history of celestial mechanics in the past century in order to identify new paths to explore in the forthcoming years. It goes without saying that we possess neither sufficient knowledge nor experience to thoroughly review the celestial mechanics of the past century. Neither can we predict its way in the new century. What is written in this manuscript is our recognition and anticipation of the past and the future of several specific areas of celestial mechanics in which we are interested. In this sense, our description may be biased toward the authors' favor. Long before our attempt, V. Szebehely (1997) stated *Open problems on the eve of the next millennium.* Though the scope of our manuscript is not as large as the next millennium, we hope our attempt serves as a milestone for getting clues and opening new frontiers in celestial mechanics.

Major flows in nineteenth-century celestial mechanics converged at H. Poincaré. And major flows in twentieth century celestial mechanics diverged out from G. D. Birkhoff. The stability analysis by A. M. Lyapunov, which he started in the late nineteenth century, also grew into a large flow in the early twentieth century. These three great masters established many of the fundamental concepts of modern celestial mechanics—integrability, convergence of perturbative developments, normal forms, stability of dynamical systems, and so on. In the 1940s, C. L. Siegel worked on the singularity of the three-body and small denominator problems. Based on what these researchers had established, the research of dynamical systems in the 1950s and the 1960s produced the greatest achievement of celestial mechanics in the twentieth century: the Kolmogorov–Arnold–Moser (KAM) theory. Later this research flow morphed into the investigation of chaos phenomena, thanks to the development of computer technology.

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In the first half of the twentieth century, after Poincaré and Birkhoff, celestial mechanics flourished in a variety of forms. The understanding of planetary motion in our solar system progressed, led by the development of accurate perturbation theories. The progress of planetary perturbation theories stimulated the search for the ninth planet, resulting in the discovery of Pluto by C. W. Tombaugh in 1930. Research on major and minor planets alike became one of the central issues of celestial mechanics in this era as astrometric observation became increasingly precise. Motivated by the identification of asteroid families by K. Hirayama in 1918, the quantitative research of asteroid dynamics began.

The Soviet Union's launch of the world's first artificial satellite, Sputnik in 1957, turned out to be one of the most significant stimuli for celestial mechanics. This epochmaking event marked the start of the space age, and owed much to the development of advanced technology. From the viewpoint of celestial mechanics, Sputnik represented the triumph of Newtonian mechanics and perturbation theories. Theoretical studies of the motion of artificial satellites made great advances after Sputnik's launch.

As well as to many other fields of science and engineering, the development of fast computers has made an enormous contribution to celestial mechanics, replacing the numerical studies by hands ("human" computers) that had already started in the early twentieth century. The use of electronic digital computers for numerical studies in celestial mechanics began in the 1950s. At first, the general three-body problem was targeted as an example study using electronic computers. Regularization methods that had been initially devised for theoretical purposes have become important since the 1960s. The techniques of regularization partially overcame one of the biggest difficulties in celestial mechanics: collisions. Computers are also used as instruments for purely theoretical research. For example, developments of dynamical theories were enhanced a great deal owing to the detailed numerical survey of phase space.

The progress of numerical study in celestial mechanics owes a lot not only to the development of computer hardware, but also to sophisticated numerical algorithms. A variety of numerical integration schemes such as the Runge–Kutta methods, various predictor–correctors, the extrapolation method, and the family of symmetric integration schemes has contributed greatly to the progress of numerical celestial mechanics. In the 1990s, the knowledge of these numerical techniques was combined with that of Hamiltonian dynamics, resulting in a totally new type of numerical scheme: symplectic integrators.

Digital computer technology yielded huge innovations not only in the field of theoretical computation but also in the area of advanced engineering of astronomical observation that has given celestial mechanics a substantial stimulation. For example, detailed observations of solar system objects such as asteroids, comets, and the trans-Neptunian objects including the Kuiper Belt and the Oort Cloud objects have helped solar system dynamics achieve a huge progress. A number of spacecrafts with cutting-edge digital technology have explored the solar system objects, giving us precious information about the dynamical nature of our planetary system. In addition to this, the recent rush of discoveries of extrasolar planetary systems, owing to the development of new observing technologies, has significantly extended the field of celestial mechanics since the mid 1990s.

Bearing this background in mind, from the next section we review some of the major achievements and remaining problems of twentieth century celestial mechanics [1]. We have categorized the major fields of celestial mechanics into four parts: dynamical systems (Section 2), the three-body problem (Section 3), solar system dynamics (Section 4), and numerical methods (Section 5). Section 6 presents some discussion and concluding remarks. We are fully aware of our inability to describe several important topics such as relativistic celestial mechanics, boundary area to stellar dynamics, current progress of artificial satellite theories, observation-related research of near-Earth objects, detailed description about ring-satellite dynamics, and many other fields. This ignorance just reflects the limitation of our abilities and experience in these fields. We have noted major missing topics and their importance at the beginning of each section. Throughout this manuscript, we would like to provide some thoughts and questions for our readers: What is celestial mechanics? How should we define it? Since the progress of celestial mechanics is quite rapid, the answer to these questions might not be straightforward. But the subtleties of such questions could help us to delineate new academic frontiers in the new century.

2. Dynamical systems—from Poincaré to chaos

Towards the end of the nineteenth century, H. Poincaré revolutionized the human view of nature (Poincaré, 1892). Nature is not so simple as the simple form of the Newtonian equations of motion suggests, something Poincaré realized through his own experience. His first experience perhaps was his failure to show the solvability of the restricted three-body problem in the Grand prix paper for the Swedish prize in 1889. The failure was said to be related to the stable and unstable manifolds of periodic points. At first, Poincaré implicitly assumed that these manifolds were connected smoothly. But later, he realized that they generally intersect transversely. His second experience came with his success in proving the analytic non-integrability of the restricted three-body problem in 1890. His failures and successes might have made Poincaré a modest person. After these experiences, Poincaré proposed to study the general

[1] Note that our definition of the border between the twentieth and twenty-first century is somewhat ambiguous. Sometimes we consider the publications from 2001 to today as those of the "previous century". Basically we regard research undertaken now and in the near future as that of the "new century".

problem of dynamics with a Hamiltonian F, developed in terms of the powers of a small parameter μ in the following manner:

$$F = F_0(p) + \mu F_1(p, q) + \mu^2 F_2(p, q) + \cdots, \tag{1}$$

where p and q are the action–angle variables, F_0 is the unperturbed part of F depending only on p, and $F_1, F_2, \ldots$ are periodic functions of period 2π with respect to q. This problem became known as the fundamental problem of dynamics. The moral of the fundamental problem is as follows: Do not swim far from safe land (i.e. the integrable system); if one ignores this warning, one may be swept up in the retreating flow and drowned in the vast sea of the unknown. However, an infinite sequence of perturbative transformations and an infinite series of perturbative developments do not converge due to the existence of small denominators. Any perturbation theory would not yield exact solutions, no matter how high its order is. If an infinite series developed in powers of small parameters converges, it would mean the dynamical system under consideration is integrable. A common understanding at the end of the nineteenth century was that the general problems of celestial mechanics are non-integrable. Integrable systems such as a special spinning top are very rare.

Poincaré may have sensed the limitations of differential equations, or the difficulties of solving differential equations. Newton's laws are differential, whereas Kepler's laws are integral. Differential laws describe the relations in nature locally; they connect the relations of physical quantities spatially and temporally in the close vicinity. On the other hand, Kepler's laws describe the global relations between physical quantities. Natural laws might be described using procedures other than differential equations. Nevertheless, according to Daniel L. Goroff who is the editor of the English translation of Poincaré (1892), Poincaré appreciated the great conceptual power of differential equations. According to Poincaré, thanks to the postulate that the actual state of the world depends only on the most recent past, without being directly influenced by the memory of the distant past, one can start from writing out differential equations.

Poincaré kept his distance from pure differential equations; he undertook the qualitative and geometrical study of dynamical systems. Poincaré surfaces and Poincaré maps are typical examples of this direction in his research. He was interested in periodic orbits as phenomena forming the backbone of phase space of dynamical systems. He found doubly asymptotic solutions. These are generally transverse intersections of singly asymptotic manifolds of solutions to periodic solutions, solutions which Poincaré called "homoclinic" or "heteroclinic". These solutions now play central roles in the study of dynamical systems. He was also interested in the stability of the solar system. Much of Poincaré's work other than in the field of celestial mechanics, such as on triangulation, homology, torsion, and the fundamental group in algebraic topology, was "motivated by Poincaré's desire to extend his ideas about dynamics to higher dimensions," again according to Daniel L. Goroff.

Virtually independently in Russia, motivated by the stability problem of planetary motion, A. M. Lyapunov developed a stability theory of dynamical systems (Lyapunov, 1892). Lyapunov's first method is related to the stability analysis of equilibrium points of non-linear systems, in which equilibrium points are linearly asymptotically stable. But when the linear stability of the equilibrium point is marginal, his first method is not applicable. So, Lyapunov proposed his second method for this case. One looks for a Lyapunov function which is positive and whose time derivative is zero or negative. The equilibrium point is stable if the time derivative of the Lyapunov function is zero, while the point is asymptotically stable if the derivative is negative. Lyapunov's results had a profound influence on the celestial mechanics of the twentieth century. We recognize this fact by referring to the notions named after Lyapunov, such as the Lyapunov characteristic exponent, the Lyapunov function, Lyapunov's first and second methods, the Lyapunov stability, and the Lyapunov asymptotic stability. Actually, equilibrium points in Hamiltonian systems are always marginal, and the time derivative of the desired Lyapunov function is zero, i.e., the Lyapunov function is the integral of motion. Thus, the search for Lyapunov functions is equivalent to the search for new integrals of motion.

G. D. Birkhoff began his substantial scientific career by proving Poincaré's last geometric theorem on the existence of fixed points for twist maps. Then following Poincaré, Birkhoff qualitatively studied the motion of the third body in the restricted three-body problem (Birkhoff, 1915). He returned to this subject several times in the course of his life. He introduced various concepts in general dynamical systems that are extensively used at present, such as α- and ω-limit points, minimal motions, central motions, wandering and non-wandering points. Formally, he obtained the normal form of great generality for Hamiltonian and Pfaffian systems (Morse, 1946). Though many of these notions are valid for an arbitrary degree of freedom, Birkhoff's analysis was restricted to systems with two degrees of freedom. The analysis of two-dimensional area-preserving maps, which was started by Poincaré and deepened by Birkhoff, is extremely popular at present. Even when we cannot follow the individual orbits of celestial bodies, we are able to follow a group of orbits, or a bunch of orbits. This was Birkhoff's basic idea which commemorated the birth of the research into dynamical systems.

Many researchers have followed the legacy of Poincaré, Birkhoff and Lyapunov. C. L. Siegel was one of them. His most important contribution to the study of dynamical systems is twofold: one aspect is the solution of the small denominator problem in a simpler form. The second is the proof of general divergence of the normal form procedure proposed by Birkhoff (Siegel and Moser, 1971). The

KAM theory (see Section 2.1) started from a brief note by A. N. Kolmogorov (1954) where the proof of the KAM theorem was only outlined with the use of Newton's method to obtain zeros of a function. V. I. Arnold (1963) proved the theorem for the analytical Hamiltonian systems (see Section 2.1), and J. K. Moser (1962) proved the theorem for smooth 2-dimensional twist maps (see Section 2.3). S. Smale (1967) devised the so-called horseshoe in an illustrative manner. M. Hénon (1976) proposed a map, now called the "Hénon map" which may exhibit strange attractors. Benedicks and Carleson (1991) proved that the Hénon map has a strange attractor for some parameters. Strange attractors have been found numerically by Lorenz (1963) and Ueda *et al.* (1973), eventually leading to the world of chaos.

In the following subsections, we describe major four issues that derived from the work of Poincaré, Lyapunov, and Birkhoff. The first of these is the KAM theory including the Arnold diffusion and the Aubry–Mather sets. The second is the world of chaos. The third is the area-preserving twist maps derived from Hamiltonian systems. The final issue is integrability. Major important topics of this field that are not described in this section include integrability of partial differential equations and advance of canonical perturbation theories (though we will briefly mention some perturbation theories in relation to the solar system dynamics in Section 4.1). Readers can study more about these topics from Brouwer and Clemence (1961), Hagihara (1970, 1972a,b), Novikov (1981), Soffel (1989), Brumberg (1991), Arnold *et al.* (1993), or Boccaletti and Pucacco (1996, 1998).

2.1. *The KAM theory*

Pierre Simon de Laplace, who lived in the latter half of the eighteenth century and in the first quarter of the nineteenth century, imagined the so-called "Daemon of Laplace" in his book (Laplace, 1814). Literally, Laplace wrote "Give me all the initial conditions. Then I will predict as precisely as possible the future of the world," expressing the victory of Newtonian mechanics. Celestial mechanists, scientists, and the general audience of the nineteenth century anticipated or wished the world to be precisely predictable. But at the same time, strange phenomena which could not be explained by simple mechanics were also being discussed—thermal and electromagnetic phenomena. These gradually became the subjects of modern sciences. Confidence in the mechanical world was shaken like an unstable sky scraper. Eventually, Poincaré proved that perturbative expansion does not converge in general. Almost at the same time, came the emergence of statistical mechanics. The pendulum of the world seemed to swing toward extreme ambiguity, and suddenly, the future looked statistically uncertain.

However, we have strong counterevidence to the idea that the future is stochastic: the longevity of the solar system (see also Section 4.2). The lifetime of the solar system that we know has grown along with our understanding of the universe. In the nineteenth century, our universe was supposed to be only several tens of million years old. Even in the 1920s, the age of the universe was considered to be as young as two billion years. In the 1950s, the age of the solar system turned out to be about 4.5 billion years old from the results of radioisotope analysis of meteorites. This finally indicated the possibility that planetary motion has been very stable for billions of years since its formation, maintaining a nearly regular (in the terminology of Hamiltonian dynamics, "quasi-periodic") motion. On the other hand, the equations of motion describing the solar system are non-integrable, and planetary motions can be, and actually are, chaotic (see Section 2.2). If the motion of the planets has been bounded in a small region of the phase space for billions of years, we may expect that there exist barriers that prevent planets from wandering out of their accustomed place. Is this expectation true?

Here we have the very theory that answers precisely to this question: the KAM theory (Kolmogorov, 1954; Arnold, 1963; Moser, 1962). According to the KAM theory, when non-integrability is weak, almost all of phase space is occupied by such barriers, i.e., invariant sets of points that are filled with quasi-periodic orbits. As the system goes further from integrable state, these barriers become sparse. These barriers are formed by quasi-periodic motions. Here, a quasi-periodic motion is a motion described by a quasi-periodic function of time:

$$x(t) = f(\omega_1 t, \ldots, \omega_k t), \tag{2}$$

with frequencies $\omega_1, \ldots, \omega_k$ that are rationally independent. All other motions in phase space are sandwiched by these quasi-periodic motions. This is topologically true in two degrees of freedom case since quasi-periodic motions form two-dimensional torus which divides three-dimensional energy surface into two parts. The solar system planetary motion might not be described by (2). However, planetary orbits will not go beyond the barriers whose existence is assured by the KAM theory. Again, the pendulum swung to the opposite side. "Okay, our future is not so uncertain." Not only to celestial mechanics, but to nearly all fields of modern mathematical sciences, the KAM theory was a relief.

Let us briefly explain the proof of the KAM theorem along with Arnold (1963). In his paper, after showing the preparatory results from celestial mechanics and mathematics, Arnold explains the classical perturbation theory of celestial mechanics. The perturbation theory has two essential difficulties: One is the so-called small denominator problem which comes from resonance and whose existence is dense in the frequency domain. The other is the divergence of the infinite sequence of canonical transformations to drive off the perturbation to higher and higher orders. Then, Arnold introduces a program proposed by Kolmogorov (1954) to circumvent these difficulties, as well as to show the stability of fixed points and periodic orbits of near integrable systems.

Kolmogorov's (1954) program is twofold: In order to overcome the two difficulties that we mentioned above, at first Kolmogorov recommends that we be satisfied with looking at particular frequencies, and suggests that we look for tori where the quasi-periodic motions with these frequencies persist. Kolmogorov's second recommendation is to restrict the transformation to the neighborhood of such tori, avoiding resonant regions. As a result, new perturbations will be quadratically small compared with the starting perturbation. Repeating this transformation, we will obtain a smaller region of applicability of the transformation in phase space. Finally we will arrive at invariant tori. The convergence will be very fast, like popular Newton's method for obtaining the zeros of functions.

Arnold followed Kolmogorov's suggestion and carried out the program. Actually, Arnold did more. He was interested in the stability of the planetary system. In this case, the situation is not so simple: He needed to treat a degenerate problem. The integrable Hamiltonian is "properly degenerate", that is, the number of angle variables or the number of independent frequencies is less than the number of degrees of freedom. In addition, perturbed systems have fast and slow variables, i.e. two different time scales. The variables representing the orbital period are the fast ones, whereas the variables that do not appear in the integrable case are the slow ones. A typical example of fast variables in the Keplerian motion is mean anomaly, and a typical example of slow variables is argument of perihelion. In order to treat them, Arnold made precise the notion of adiabatic invariant. He divided the variables into fast and slow ones and carried out averaging over the fast variables, which is a standard technique in modern celestial mechanics.

Arnold's system has the Hamiltonian H with n degrees of freedom:

$$H(p,q) = H_0(p_0) + \mu H_1(p,q) + O(\mu^2), \tag{3}$$

where H_0 represents the integrable part, μ a small parameter, p and q canonical momenta and coordinates, part of which being action–angle variables. The last term on the right-hand side of (3) is the higher-order terms in μ. We should notice that p_0 is the action variables corresponding to the fast angular variables, and its dimension is less than n. Hence the system is properly degenerate. However, as an independent Hamiltonian, $H_0(p_0)$ is assumed to be nondegenerate. $H_1(p,q)$ is assumed to consist of three terms of different characters representing perturbations. The analytic Hamiltonian $H(p,q)$ is defined explicitly in a complex domain. This is essential in the following discussion. The assertion is that the real part of the domain of the definition of the Hamiltonian is divided into two sets: One of them is invariant with respect to the canonical equations with this Hamiltonian, and the other is small. The larger set consists of invariant n-dimensional analytic tori which are little deformed from the integrable ones. Motions on these tori are quasi-periodic.

In order to prove this fundamental theorem, Arnold first carried out the integration of the Hamiltonian over the fast variables, changing slow variables into the polar coordinates and reducing the original Hamiltonian to the form

$$H(p,q) = H_0(p_0) + H_1(p_0,p_1,q_1) + H_2(p_0,p_1,q_0,q_1), \tag{4}$$

where $\dim p_0 = n_0$ and $\dim p_1 = n_1$ with $n_0 + n_1 = n$, and the suffix of H indicates the order of the term. H is defined in a complex domain F as

$$F = \{p_0 \in G_0, |\mathrm{Im}\ q_0| \le \rho, p_1^2 + q_1^2 \le R\}. \tag{5}$$

where G_0 is a domain, and ρ and R are positive numbers. Then, Arnold constructed an inductive scheme with an inductive theorem, an inductive lemma, and a fundamental lemma accompanied by a series of technical lemmas. He applied this scheme to the Hamiltonian (4). His purpose was, through an infinite sequence of transformations, to nullify the H_2 term. He showed in the fundamental lemma that there is a canonical transformation such that in the new Hamiltonian, the new H_2 is quadratically small compared with the original H_2, though the domain of the Hamiltonian becomes smaller with certain resonant domains excluded from the original domain. Using the fundamental lemma, he proved the inductive lemma that roughly states that the Hamiltonian system of the form of (4), which is slightly different in Arnold's original paper, can be transformed into a new Hamiltonian system. Arnold also proved that the new system possesses a form similar to that of the original Hamiltonian with a quadratically small H_2 term and with a reduced domain. Then in the inductive theorem, he repeated an arbitrary number of times the transformation defined in the inductive lemma, and reduced the absolute value of the H_2 term to an arbitrarily high order. And finally, Arnold went to the limit and proved that the phase space is divided into two parts: One of them occupies almost the entire phase space and is filled with tori, and the other occupies a small set in the sense of measure.

Some serious questions remained open after the proof of the KAM theorem. A big one was that of the higher dimensional case. n-dimensional tori of quasi-periodic motion do not divide the $2n$-dimensional phase space. Therefore, tori are not the barriers in the higher dimensional cases. General motion between two tori is not confined, and the solution gradually diffuses. This phenomenon is called the Arnold diffusion (Arnold, 1964; Nekhoroshev, 1977). How fast is the diffusion, then? Could the solar system disintegrate soon? Fortunately, the diffusion speed is slow. The diffusion proceeds along the web of stable and unstable manifolds from various periodic orbits. For the Hamiltonian

$$H_0(p) + \varepsilon H_1(p, q), \tag{6}$$

with action p and angle q with a small parameter ε, there exist constants a, b such that for

$$0 < t < T = \frac{1}{\varepsilon} \exp\left(\frac{1}{\varepsilon^a}\right), \tag{7}$$

the action satisfies

$$|p(t) - p(0)| < \varepsilon^b. \tag{8}$$

The diffusion velocity should depend on the distance d from the invariant torus. More precisely (Morbidelli and Giorgilli, 1995), the diffusion is super-exponentially slow ($\sim 1/\exp[\exp(1/d)]$) until some threshold distance. The diffusion is exponentially slow ($\sim 1/\exp(1/d)$) until the next threshold distance. And, it is quadratically slow ($\sim d^2$) until the final threshold distance (Morbidelli and Guzzo, 1997). The last threshold is the boundary to the sea of strong chaos.

An ideal concept like the Arnold diffusion and its confirmation are always two different things. Xia (1992) claimed to have confirmed the existence of the Arnold diffusion in the planar isosceles three-body problem. However, in a realistic dynamical system with a large number of degrees of freedom, nobody can say for sure that he/she has observed the Arnold diffusion yet. For us to "see" the Arnold diffusion, we need to take a special care to somehow shorten its timescale. Minor bodies in our solar system might give us a clue to "see" the Arnold diffusion, because the dynamical lifetime of some minor objects could be so short that the timescale of their Arnold diffusion might be observable in numerical experiments. Some such trials have already started. For example, Morbidelli (1997) proposed using asteroids with short dynamical lifetimes in order to observe the Arnold diffusion in numerical experiments. Morbidelli and Guzzo (1997) explored the dynamical structure of the 2:3 mean motion resonance with Neptune in the Kuiper Belt objects using numerical integrations. They focused on the existence of slowly diffusing chaotic orbits that escape from the 2:3 resonance after billions of years, which might be a part of the Arnold diffusion. In any case, we need a very large amount of computation with a very high accuracy (hopefully with a large-number of arithmetic digits) to observe the Arnold diffusion within a realistic timescale.

Before the end of this subsection, let us give a remark on an interesting application of Nekhoroshev's spirit to the stability of asteroid orbits (Giorgilli and Skokos, 1997). This work shows that the Nekhoroshev-type estimate is applicable not only to nearly integrable systems but also to systems near an equilibrium using the Birkhoff normal form. The essence of the idea is contained in the trivial inequality

$$|p_j(t) - p_j(0)| \leq |t| \sup_{\text{domain } \Delta} |\dot{p}_j| \tag{9}$$

where p_j is the action variable under consideration, t is time, $\dot{p}_j$ is the time derivative of p_j, and Δ is a domain where the diffusion takes place. We transform the expanded Hamiltonian to a normal form up to an optimal order r as

$$H = \sum_{k=2}^{r} H_k + \mathcal{R}^{r+1} \tag{10}$$

where H_r is a function only of p_j $(j = 1, \dots, n)$ of order $r/2$, and $\mathcal{R}^{r+1}$ is a remainder of higher orders. Then, we obtain

$$\dot{p}_j = \{p_j, H\} = \{p_j, \mathcal{R}^{r+1}\}. \tag{11}$$

Giorgilli and Skokos (1997) succeeded in estimating the supremum of the norm $||\{p_j, \mathcal{R}^{r+1}\}||$ in a neighborhood of L_4 in the restricted three-body problem, and discussed the stability of actual Jupiter's Trojan asteroids.

What happens if quasi-periodic motions no longer constitute a surface? How would it look like when the torus of quasi-periodic motion gets disintegrated? This question had been open since the dawn of KAM theory research, and was solved for two-dimensional twist maps by Aubry and LeDaeron (1983) and Mather (1982). A KAM curve, when it disintegrates like a sweater with its weft lost, becomes loose and wavy. When we look at the surface of section of the KAM curve, a continuous curve is disintegrated into a Cantor-like discontinuous set: an Aubry–Mather set. Now we are in the world of chaos. We will come back to the Aubry–Mather theory in the section on twist maps (Section 2.3).

2.2. *Chaos*

Chaos was recognized by the ancient Chinese philosopher, Zhuangzi, in the 4th century BC. Zhuangzi realized that chaos was essential in the world, and that the major part of the world could become meaningless, dark, or even dead without chaos. Mathematical chaos has its origin in the work of Poincaré. Poincaré carefully observed special dynamical flows going into and emanating from hyperbolic fixed points of the restricted three-body problem. He also observed complicated tangles of stable and unstable manifolds. Poincaré shuddered at the glimpse of infinity that Newtonian dynamics let him see. Just like a baker's dough, our world is folded and extended.

Later in the twentieth century, the existence of chaos has been confirmed independently and concurrently by many authors in various fields: Lorenz (1963) in meteorology, May (1976) in ecology, Ueda *et al.* (1973) and Ueda (1979) in electric circuits, Li and Yorke (1975) and Feigenbaum (1978) in one-dimensional maps, and Ruelle and Takens (1971) in turbulent flows. The so-called "fractal" of Mandelbrot (Mandelbrot, 1977) turned out to underlie these phenomena, and their relation led to the popularization of chaos. Later it turned out that Poincaré, Birkhoff, and Smale were observing the same phenomenon years before.

The first astonishment among physicists concerned with chaos came from the observation that phenomena of random nature were visible in systems with low degrees of freedom. In fact, Landau and Lifshitz (1959) proposed expressing the motion of fluid by a quasi-periodic function of time (2). Landau and Lifshitz (1959) presumed that the number of frequency k would increase as the fluid approached turbulence; the degree of freedom of a dynamical system would increase as the flow became laminar to turbulent. They expected that unpredictable motion would appear only in a system with large degrees of freedom. Full turbulence should have an infinite degree of freedom. The reality is, however, even a system described by three ordinary differential equations exhibits chaos. Chaos appears in a Hamiltonian system with only two degrees of freedom. Interestingly, larger astonishment for celestial mechanists arose from the fact that chaos was not restricted to Hamiltonian systems. Chaos is also seen in dissipative systems.

The essence of chaos is folding and extending. Its simplest representation was given by Smale (1967), a horseshoe: Extend a square, fold it in the middle, and put it on the original square. This model provides us a glimpse into chaos. A chaotic system is exponentially unstable (i.e. coming from extension), but it comes back close to its original location (i.e. folding). Repetition of folding and extending forces the system to forget the memory of its initial conditions. Hence a chaotic system has a very high sensitivity to initial conditions. Now, even the Laplace daemon gets seriously annoyed, faced with infinite repetitions. Is there any point in predicting the future of a chaotic system? The intricate and complicated structure of dynamical systems that Poincaré did not want to describe troubles scientists again. The complexity of chaos is ubiquitous in every scale of our universe.

As the system becomes chaotic and loses regularity, KAM tori eventually disintegrate. The final motion (i.e. the states when time $t \to \infty$) of a chaotic system depends sensitively on its initial conditions. The initial value space that is mapped according to the different types of final motions shows us fractals. This fractal figure bifurcates and changes its features when the external parameters change. Moreover, this bifurcation itself takes place chaotically. The remnant of the KAM curve is a Cantor set. It takes a long time for a phase point to move through the holes of the Cantor set. In a Hamiltonian system with two degrees of freedom, when the last KAM curve disintegrates, a phase point so far confined within a bounded region is now set free to move toward infinity. However, according to numerical integrations, it is a slow process just after the disintegration. One evidence of the slowness is the difficulty of the determination of the parameter value when the last KAM curve disintegrates. See Fig. 5.3 of Chirikov (1979). In relation to the disintegration of KAM curves, Hénon and Heiles (1964) numerically discovered Hamiltonian chaos by observing the disintegration of KAM curves. Hénon (1965) discussed the same phenomenon using the restricted three-body problem as an example. He devised a planar quadratic map which exhibits a strange attractor (Hénon, 1976). This map is now called the Hénon map.

Chaos is the ubiquitous and normal state of the universe. But this was not common sense until the final decade of the twentieth century when the efforts of scientists began to reveal chaotic phenomena in a variety of fields. Examples of chaos in our solar system were extensively dug up and exposed to the light of science by Jack Wisdom from the viewpoint of the Lyapunov exponent (Wisdom, 1982, 1983, 1987a,b; Wisdom *et al.*, 1984). Wisdom dissertated about chaos in the motion of small bodies in the solar system such as asteroids and a saturnian satellite, Hyperion. He also found chaos in the orbital motion of major planets. Solar system chaos is extensively reviewed in Lissauer (1999).

Let us briefly introduce the definition of the Lyapunov exponent. First, consider a one-dimensional discrete dynamical system denoted by f. Then, Lyapunov exponent $\lambda(x)$ of the orbit of x is defined by

$$\lambda(x) = \lim_{k\to\infty} \frac{1}{k} \ln \left| \frac{df^k(x)}{dx} \right|. \tag{12}$$

$\lambda(x)$ represents the growth of the infinitesimal difference of the initial conditions in phase space as a function of the number of iteration, k. If the growth is slower than an exponential one, the Lyapunov exponent is zero. Once a point x is given, then the Lyapunov exponent is unique, i.e., any point $f^j(x)$ on the orbit has the same Lyapunov exponent. So the notion of the Lyapunov exponent corresponds to a trajectory in phase space. For an n-dimensional discrete dynamical system, there are n Lyapunov exponents. We usually deal with the largest Lyapunov exponent.

The characteristic time of a chaotic system is often measured in the Lyapunov time (cf. Benettin *et al.*, 1976). The Lyapunov time is defined as the inverse of the maximum Lyapunov exponent. By definition, the Lyapunov time is an average e-folding time of the separation of two different trajectories. The series of work by J. Wisdom, just mentioned, extensively used Lyapunov exponents and Lyapunov time to render visible the degree of chaos in solar system dynamics. Sussman and Wisdom (1988) found that the orbital motion of outer planets including Pluto was chaotic, judging from the largest Lyapunov exponent in their long-term numerical integration. Sussman and Wisdom (1992) also discovered that the Lyapunov time of the motions of the four inner planets is only a few million years, indicating a typical chaotic system. Their conclusions were later confirmed by an accurate semi-analytical perturbation theory of J. Laskar (Laskar, 1989, 1990, 1994, 1996; Laskar *et al.*, 1992).

However, a short Lyapunov time does not mean that a chaotic system becomes unstable and disintegrates very quickly; the Lyapunov time is different from the global instability time of a dynamical system. One of the possible explanations for this fact is that the Lyapunov exponent is a local notion. Exponential separation, when it ever happens, takes place in a linear regime: two trajectories in phase space separate from each other exponentially in time, starting from their infinitesimally small distance. Strictly speaking, we do not know what happens when the separation becomes macroscopically large. As we mentioned, the essence of chaos is folding and extending. The positive Lyapunov exponent implies extending. But to which extent? The folding process may bring two trajectories back to the initial small separation.

A lot of evidence indicates that the planets in our solar system have survived nearly 4.6 billion years, far longer than their Lyapunov time, with a global stability (e.g. Ito and Tanikawa, 2002). This typically shows that Lyapunov time T_L does not help us very much in predicting the real instability time T_I of the system. There have been, however, some efforts to understand the relationship between T_L and T_I, utilizing the dynamical behavior of solar system minor bodies. For example, some asteroids go into unstable orbits over only a few hundred to a few thousand years, so these objects might be used for determining the relationship between T_L and T_I (Lecar and Franklin, 1992; Lecar *et al.*, 1992). Lecar *et al.* (1992) proposed a relation

$$T_I \propto T_L^{\gamma}, \tag{13}$$

through a variety of numerical experiments for asteroids with unstable orbits. The exponent γ looked close to 1.8 in their experiments. But the dynamical meaning of either relationship (13) or the value of $\gamma \sim 1.8$ is not clear yet, though some qualitative explanations exist (e.g. Morbidelli and Froeschlé, 1996). If relationship (13) holds true in many cases, a numerical integration of the length of Lyapunov time T_L of the system would suffice to tell us the real instability time T_I, which could save us computer time.

A chaos that does not cause any real or global instability over a much longer time than the Lyapunov time is often dubbed a "stable chaos" or "weak chaos" (e.g. Froeschlé *et al.*, 1997; Šidlichovský, 1999; Tsiganis *et al.*, 2002). For example, many asteroids are dynamically stable for much longer than their Lyapunov timescales. This kind of stable chaos in asteroid dynamics is relevant to high-order mean motion resonances with Jupiter in combination with secular perturbations on the perihelia of the asteroids. These perturbations move the asteroid orbit from one high-order resonance to another in some irregular ways (e.g. Milani and Nobili, 1997). Chaos in asteroidal orbital motion has also been extensively studied by N. Murray and M. Holman (Murray and Holman, 1997, 1999; Murray *et al.*, 1998). Murray and Holman (1997), in particular, presented an analytic theory of asteroid motion near resonances in the planar elliptic restricted three-body problem. Their theory predicts the location and extent in the semimajor axis and eccentricity space of the chaotic motion, the Lyapunov time, and the time for objects on chaotic orbits to be removed from the system. Their theory predicts that asteroids in a number of high-order mean motion resonances possess very short Lyapunov times, such as 10^5 years, but with instability times as long as the lifetime of our solar system. Incidentally, we should notice that apparent chaos in solar system dynamics sometimes comes out as a result of numerical error or instability (cf. Rauch and Holman, 1999; Ito and Kojima, 2005).

Before closing this subsection, we would like to mention a bit about classification of the degrees of chaos in Hamiltonian systems. Hamiltonian systems have neither attractors (i.e. sinks) nor repellers (i.e. sources). Phase points straggle out almost without purpose in the phase space. Everywhere is filled with complicated structure. When points approach the local KAM tori, they are trapped for a long time in a stagnant region. It seems difficult to obtain an average time for a point to stay at a particular place (or region) in phase space. This structure becomes more and more entangled as the system goes further from the integrable state. Strange as it might sound, there appears a random system as a limit of this complexity, and the average staying time of a point can be calculated.

The complexity is measured by topological entropy and/or measure entropy. The topological entropy is a measure of maximum complexity of a system, whereas the measure entropy is the average complexity over phase space. The definition of topological entropy is not so straightforward. For simplicity, consider a discrete dynamical system f of a compact phase space X. In general, two orbits in a dynamical system separate with time. Let $B(x, \varepsilon, n)$ be a set of points y such that the largest of $d(f^i(x), f^i(y))$ $(i = 0, \ldots, n-1)$ is less than ε. Here $f^i(\cdot)$ is the i-th iterate of the map, and $d(\cdot,\cdot)$ is a distance function. Let $N(\varepsilon, n)$ be the minimum number of $B(x, \varepsilon, n)$ to cover X. Then, topological entropy $h_{\mathrm{top}}(f)$ is defined as

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \left[\limsup_{n} \frac{1}{n} \ln N(\varepsilon, n) \right]. \tag{14}$$

Suppose that we have a finite resolution ε. We can not recognize two different states as "different" when their separation is smaller than ε. We can roughly say that $N(\varepsilon, n)$ is the number of states we can recognize after the n-th iteration. Then $\frac{1}{n} \ln N(\varepsilon, n)$ is the exponential increase (per iteration) of the number of distinguishable states. Finally, $h_{\mathrm{top}}(f)$ is the asymptotic value of information production rate per iteration for infinitesimal resolution.

Both of the entropies, particularly measure entropy, are very difficult to calculate. Some methods have been developed for estimating the lower bound of the topological entropy for a given system. The simplest way is to find a horseshoe. For example, the lower bound is ln 2 if the Smale horseshoe exists (cf. Section 15.2 of Katok and Hasselblatt (1995)). Another method is to find a periodic orbit from which a non-trivial braid can be constructed. In the case of the Hénon map, the lower bound of the topological entropy is measured by the increasing rate of the number of periodic orbits per period (cf. Sterling *et al.*, 1999).

In general, the larger the structure, the more responsible shorter periodic solution is. Here, by the structure we mean the distribution of the so-called islands and sea in the phase space. Some of the bifurcated solutions are unstable from the start, and others are initially stable and become unstable later. Stable and unstable manifolds emanating from the unstable solutions intersect each other, as well as other unstable and stable manifolds from other unstable solutions. They together make a complicated network of invariant manifolds. Phase points move along this network. The qualitative nature of the network differs from system to system. The network of the phase space of the solar system and that of a simple three-body problem are its examples. Our principal goal may be to clarify the structure of the network of individual systems. A global network connected to infinity leads to the instability of the system. In systems with degrees of freedom larger than two, any phase point seems to be close to this global network. This is the origin of the Arnold diffusion that leads to instability.

2.3. *Twist maps*

There is a direction of studies in which some two-dimensional maps of a Hamiltonian system with two degrees of freedom, in particular twist maps, are used to investigate the start, development, and degree of chaos. The study of twist maps again goes back to Poincaré. He considered a theory of consequents: a theory of the Poincaré surface and Poincaré map in the present terminology. The example object of the study was the restricted three-body problem. Observing periodic orbits of the problem, Poincaré took the Poincaré map of its neighbor. If neighboring points move on a closed curve around the fixed point, then the fixed point is stable. Behavior of points near the fixed point can be viewed like that in a punctured disk, i.e. a ring-like structure when the fixed point is removed from its center and when the center is blown up. Poincaré arrived at his "last geometric theorem" (Poincaré, 1912) starting from the surface map of the periodic orbits of the restricted three-body problem. Let us describe a famous theorem known as Poincaré–Birkhoff:

Theorem 2.1 (Poincaré–Birkhoff) *Let us suppose that a continuous one-to-one transformation T takes a ring R, formed by concentric circles C_a and C_b of radii a and b respectively ($a > b > 0$), into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in the negative sense, and at the same time to preserve areas. Then there are at least two invariant points.*

This kind of map is now called a twist map because it twists a ring. Birkhoff (1913, 1925) started his substantial scientific career proving this theorem. Twist maps have been used in various contexts. One of the main reasons for this is that we can treat the maps very easily. The maps also express the dynamics of Hamiltonian systems with two degrees of freedom.

Birkhoff later made some remarkable contributions to the study of twist maps (Birkhoff, 1920). One of them was on the functional form of the invariant curves, now called KAM curves. Another was on the property of points in the instability zone sandwiched by two invariant curves. The following theorem represents an important part of Birkhoff's contribution:

Theorem 2.2 (Birkhoff) *In a small neighborhood of an elliptic point with irrational rotation number where the twist is monotone, any invariant curve enclosing the invariant point meets every radius vector through the invariant point in only one point. If the barred angle in the plane be drawn at the corresponding point, the curve lies entirely within it on either side in the vicinity of the point.*

In other words, the invariant curve encircling the fixed point is a graph of a Lipschitz function of polar angle. The following theorem describes the transverse motion of points in a zone of instability.

Theorem 2.3 (Birkhoff) *Let C', C'' be entirely distinct invariant curves forming the boundary curves of a ring of instability. Then, for any $\varepsilon > 0$ an integer N can be assigned such that a point P' exists within a distance ε of any point P of C' (or C'') which goes into a point Q' within ε of any point Q of C'' (or C') in $n < N$ iteration of T (or T').*

The next major step marked in the context of twist maps is the KAM theorem (or the KAM theory). In Section 2.1 we explained Arnold's proof (Arnold, 1963). In the context of twist maps, Moser (1962) discussed the persistence of invariant curves of monotone twist maps of the annulus

$$\theta_1 := \theta + \alpha(r) + F(r, \theta), \tag{15}$$

$$r_1 := r + G(r, \theta), \tag{16}$$

which are derived by adding small perturbations to an integrable case

$$\theta_1 := \theta + \alpha(r), \tag{17}$$

$$r_1 := r, \tag{18}$$

where

$$\frac{d\theta_1}{dr} > 0, \tag{19}$$

and F, G are small perturbations periodic in θ. Literally, given an integrable monotone twist map and smooth enough small perturbations, then invariant curves of rotation number ω satisfying Diophantine conditions

$$|2\pi p - q\omega| \geq \frac{\varepsilon}{q^{\frac{3}{2}}}, \tag{20}$$

survive the perturbation.

It is to be noted here that recently monotone twist maps are most frequently studied. So we usually abbreviate these maps simply as twist maps. Original twist maps considered by Poincaré and Birkhoff do not necessarily satisfy the monotone condition (19). Hereafter in this manuscript, "twist maps" basically denotes "monotone twist maps" unless otherwise noted.

Chirikov (1979) discussed the Arnold diffusion mainly using so-called standard maps defined on the cylinder as

$$\theta' := \theta + y + a \sin\theta, \tag{21}$$

$$y' := y + a \sin\theta, \tag{22}$$

where a is a parameter. The Arnold diffusion cannot be observed in two-dimensional maps. However, Chirikov proposed a concept called "resonance overlap" which is a lower dimensional analogue of the heteroclinic tangle of stable and unstable manifolds. Resonance overlap is the main pathway of the Arnold diffusion in higher dimensional systems (see also Section 4.4.1).

At nearly the same time as Chirikov, Greene (1979) paid attention to the disintegration of the KAM curves. According to the KAM theory, invariant curves survive longer when they are with the rotation numbers that are difficult to be approximated by rationals. Greene tried to determine the critical parameter value of KAM curves with a special type of irrational rotation number, hoping that the KAM theory would be still valid for this parameter range. Greene's tool was the residue criterion which has been waiting for a rigorous proof until now. Here the residue R of a periodic point is

$$R = \frac{2 - \lambda - \lambda^{-1}}{4}, \tag{23}$$

where λ is the eigenvalue of the periodic point. A regular saddle has $R < 0$, an elliptic point has $0 < R < 1$, and a inversion saddle has $R > 1$. His idea is that if the KAM curve is disintegrated, the nearby periodic points will be unstable. Let p_n/q_n be the convergent rationals, like a Fibonacci sequence, to irrational number ω. Then, according to Mackay (1992), the residue criterion is described as follows:

Conjecture 2.1 *Let R_n be the residue of a Birkhoff periodic point with rotation number p_n/q_n. Then*

$$\nu(\omega) = \lim_{n\to\infty} q_n^{-1} \ln |R_n| \tag{24}$$

exists, and $\nu(\omega) \le 0$ implies there is a KAM curve with rotation number ω, whereas $\nu(\omega) > 0$ implies there is none.

The golden mean rotation number $\frac{1+\sqrt{5}}{2}$ has the continued fraction expression $[1, 1, 1, \ldots]$, and is the most distant number from rationals. The last KAM curve is the one that has the golden mean rotation number. Greene (1979) estimated the critical parameter value for the destruction of the last KAM curve to be $0.971635\cdots$, setting $R = 0$ at the critical situation. Even now, this value is not fully explained theoretically.

Later, Mather (1982) as well as Aubry and LeDaeron (1983) succeeded in showing what happens if KAM curves are destroyed. Katok (1982) gave elementary proof of this. Motion on each KAM curve, if destroyed, remains as quasi-periodic, but the curve itself becomes a Cantor set. This kind of motion has already been investigated in a circle map by Denjoy (1932). A so-called "Denjoy counter-example" exists when the smoothness of the map is less. In this case, the motion with irrational rotation number on the circle does not fill the circle itself, but fills only a Cantor set. In twist maps, smoothness conditions of the maps are different from those of one-dimensional maps. KAM curves become Cantor sets even when the map is analytic. In twist maps, after the destruction of KAM curves, transverse motion across the former KAM curves is possible. The hole of the Aubry–Mather set (remnants of the KAM curve) becomes larger as the perturbation parameter grows. In deriving the Aubry–Mather theory, the classical variational principle is used. Monotone twist maps are extensively investigated in the 1980s and the early 1990s (e.g. Mather, 1984, 1985, 1991; Aubry and Abramovici, 1990; Herman, 1983, 1986; Le Calvez, 1987, 2000; Mackay and Percival, 1985; Mackay, 1992, 1993). Bangert (1988) gave an excellent review. Mather (1991) demonstrated the existence of orbits with complicated behavior in a zone of instability.

In the meantime, efforts to estimate the topological entropy of low-dimensional dynamical systems by constructing braids has begun (e.g. Matsuoka, 1983). This method has been applied to twist maps (e.g. Hall, 1984). A periodic orbit of a surface map can be represented as a braid. In fact, suppose a discrete dynamical system f is given on a disk, and f has a period-n orbit. We put the second disk below the first one separated by a distance. Let p_i and p'_i $(i = 1, 2, \ldots, n)$ be orbital points of the same periodic orbit on the disks above and below. We connect these points by strings so that p_1 is connected to p'_2, p_2 to p'_3, …, and finally p_n to p'_1. This is exactly the braid with n strings ("n-braid", in short). We call p_i and p'_i $(i = 1, \ldots, n)$ the starting and end points. We regard any two braids equivalent if they transform each other under continuous movements of starting and end points together. An equivalent class forms a braidtype. The set of braidtypes is the set of inequivalent braids. The braid is an invariant characterizing topological behavior of orbits for 2-dimensional maps. If the system becomes more chaotic, braids that are formed with newly born periodic orbits may become complicated. We have a convenient theory to estimate topological entropy of braids (Matsuoka, 1993). Thus, we can estimate topological entropy of dynamical systems by constructing braids from periodic orbits.

At the turn of the century, the study of twist maps seems to be in an interlude: Researchers seem to have lost their way. In which direction are we to go? In this opaque age, some people raise their hands and point in the direction of non-Birkhoff periodic orbits, i.e., non-monotone periodic orbits. We hope that this direction yields a promised land. Non-monotone periodic orbits or non-Birkhoff periodic orbits are the orbits that do not preserve the orbital orders.

Let us briefly explain these orbits. Consider a twist map f defined in the infinite cylinder $S^1 \times \mathbf{R}$. We lift the map to the universal cover $\mathbf{R} \times \mathbf{R}$ and denote it by $\widehat{f}$. Let $\{\widehat{f}^k(\widehat{z})\}_{k\in\mathbf{Z}}$ be the lift of orbit $\{f^k(z)\}_{k\in\mathbf{Z}}$ where $\widehat{z}$ is a lift of point z. We define the extended orbit of $\widehat{z}$ by

$$eo(\widehat{z}) = \{\widehat{f}^k(\widehat{z}) + (l, 0)\}_{k,l\in\mathbf{Z}}. \tag{25}$$

Then, the orbit of z is monotone if the following condition

$$\pi_1(\xi) < \pi_1(\zeta) \implies \pi_1(f(\xi)) < \pi_1(f(\zeta)) \tag{26}$$

is satisfied for any $\xi, \zeta \in eo(\widehat{z})$ where π_1 is the projection to the x-coordinate. If a periodic orbit is monotone, it is sometimes called "Birkhoff". If a periodic orbit is non-monotone, it is sometimes called "non-Birkhoff". In a one-parameter family of twist maps containing the integrable map, non-Birkhoff periodic points appear (or bifurcate) when the map becomes non-integrable. There are many types of non-Birkhoff orbits that represent the chaosity of the system.

Reversibility is the property of a dynamical system originated from the Hamiltonian (Zare and Tanikawa, 2002). In reversible area-preserving twist maps, we can find periodic orbits without appealing to variational principles (e.g. Tanikawa and Yamaguchi, 1987, 1989). Dynamical orders of periodic orbits in these maps are derived and lower bounds of topological entropy of the system are obtained by constructing braids from periodic orbits (e.g. Yamaguchi and Tanikawa, 2000, 2001a,c, 2002a,b,c, 2003, 2005a,b,c; Tanikawa and Yamaguchi, 2001, 2002a,b, 2005). Non-Birkhoff periodic orbits promise us a rich structure of twist maps, such as the existence of oscillatory orbits in the standard map (e.g. Yamaguchi and Tanikawa, 2004a). Exponential splitting of stable and unstable manifolds is numerically inferred from a sequence of homoclinic points on the y axis accumulating at the saddle fixed point in the standard map (e.g. Yamaguchi and Tanikawa, 2001b). Quadruply reversible non-twist maps have also been discussed (e.g. Zare and Tanikawa, 2005; Yamaguchi and Tanikawa, 2004b).

2.4. *Integrability*

The study of the integrability or non-integrability of Hamiltonian systems, as far as we understand, seems related to the formal side of celestial mechanics. As is well known, celestial mechanics started in the seventeenth century in Europe with the aim of mathematically explaining Kepler's three laws of planetary motion that were discovered at the beginning of that century. The explanation was first accomplished by I. Newton in his *Principia,* finding a force law that satisfied Kepler's laws. For the one hundred years after *Principia,* a lot of ingenious celestial mechanists and mathematicians developed differential calculus, trying to establish the laws of force on metaphysical principles: the principle of virtual velocity, the principle of d'Alembert, the principle of Maupertuis, and so on. Finally near the end of the eighteenth century, J. L. Lagrange formulated the equations named after him, which are equivalent to Newton's equations of motion, under the principle of least action (Lagrange, 1788). Lagrange gave a unified interpretation of various principles. In the nineteenth century, W. R. Hamilton, starting with the formulation of the path of light rays in geometrical optics through variational calculus, gave his Hamiltonian formalism of equations of motion, and discussed the transformation theory of dynamical systems. The existence of ignorable or cyclic coordinates motivates transformations from the original system to a system with fewer degrees of freedom. Finally, it has been recognized that the ultimate goal of the study of dynamical systems is to solve the Hamilton–Jacobi equation. This partial differential equation is for a generating function which gives the transformation from the original dynamical system to a system in which all conjugate variables are constant through motion. Thus, the formal side of mechanics has been completed in the sense that the goal of the study was revealed.

How to attain the goal, then? We do not have a systematic or automatic procedure for solving the Hamilton–Jacobi equation. We know only a few systems for which the Hamilton–Jacobi equation can be solved. One of the possible procedures for attaining the goal is the perturbative approach when the system is known to be a perturbation of an integral system. We repeat canonical transformations, and move to systems which have terms of non-integrable nature in higher and higher orders. We hope these terms to be driven away to an infinitely higher order, eventually obtaining an integrable system. This approach is not applicable to a system that is not close to any of known integrable systems. Another approach is to find additional integrals. As is easily shown, once these integrals are incorporated into the equations of motion, the degrees of freedom of the system can be reduced. At the end of the nineteenth century, Poincaré discussed various kinds of integral invariants of Hamiltonian systems. He related them to the existence of additional integrals of the system.

In the first part of Section 2 (p. 56), we already mentioned that Poincaré had succeeded in proving the analytic non-integrability of the restricted three-body problem in the late nineteenth century. More precisely, Poincaré proved the non-existence of an integral that is independent of the Hamiltonian and that can be developed in a power series of a small parameter [2] . Shortly before Poincaré, Bruns

[2] This latter condition was too restrictive. So researchers later relaxed the condition, and discovered that there can be some integrable systems for fixed perturbation parameters. For example, three-body systems are known to be integrable when they have particular mass combinations. This fact was proved through the discussions on col-

proved the algebraic non-integrability of the general three-body problem. In what follows, let us briefly review the proofs by Bruns (1887) and by Poincaré (1890), both of which are quite suggestive to us even now. The ideas underlying the two proofs differ very much.

It is not quite easy to summarize Bruns' proof of algebraic non-integrability in a few paragraphs, because it is very deep and contains a lot of steps (cf. Whittaker (1904), Chapter XIV, §164). Understanding process of this kind of proof is to dive into deep water and stay there, looking for precious stones in the depths. Here we only try to describe a rough story. Bruns first expresses formally a supposed integral as a function of coordinates and momenta, and proves that the integral must involve some of the momenta. Mutual distances are irrational functions of coordinates. Bruns introduces the sum s of mutual distances r_i, adds s to dependent variables, and tries to express integrals as an algebraic function of coordinates, momenta, and s. From this point begins the main stage of the proof. Bruns reduces the problem of finding algebraic integrals into that of finding an integral as a quotient of real polynomials of (q_i, p_i, s). Then, after a rather long argument, he shows that he can make an integral from the numerator or denominator. In fact, he expresses the numerator or denominator as $\phi = \phi_0 + \phi_1 + \phi_2 + \ldots$, where ϕ_i are homogeneous functions of momenta and ϕ_0 is the highest term in momenta. Bruns shows that if ϕ_0 does not contain s, then multiplying a rational function to ϕ, he can obtain an integral. Then he shows that ϕ_0 actually does not contain s. Here, $s = \pm r_1 \pm r_2 \pm r_3$ has generally eight different values. However, it may have the same value for special values of r_i. Bruns uses this fact to derive a contradiction. Finally, Bruns shows that ϕ_0 is a function of the momenta and the integrals of angular momentum, and that ϕ is expressible as a function of the classical integrals. It seems that the particular properties of the N-body problem used by Bruns are the weighted homogeneousness of the equations of motion, which is later used by Yoshida (1983a,b).

On the other hand, Poincaré proved generally the non-existence of an integral of the restricted three-body problem depending analytically on coordinates and momenta and also on the mass parameter. His proof proceeds as follows. Poincaré expands the supposed integral Φ as a power series in the mass parameter μ

$$\Phi = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \ldots, \tag{28}$$

where Φ_i is analytic in (q_i, p_i), and periodic in q_1 and q_2. It is to be noted that Φ itself is analytic both in (q_i, p_i) and μ. He expands the necessary and sufficient condition for the integrability, i.e., the vanishing of the Poisson bracket

$$\{H, \Phi\} = 0, \tag{29}$$

and gets two conditions

$$\{H_0, \Phi_0\} = 0, \quad \{H_1, \Phi_0\} + \{H_0, \Phi_1\} = 0, \tag{30}$$

from the lowest two orders of the expansion.

Poincaré first proves that Φ_0 is not a function of H_0 by a smart, iterative argument. Then he proves that Φ_0 includes neither q_1 nor q_2 by a non-degeneracy condition of the unperturbed Hamiltonian. This non-degeneracy was later used as a basic hypothesis in the proof of the KAM theorem. With H_0 and Φ_0 not involving coordinates, Poincaré proceeds to show that the existence of the uniform integral is not compatible with the property that Φ_0 is not a function of H_0. This problem is reduced to a commensurability problem: the commensurability condition

$$m_1 \frac{\partial H_0}{\partial p_1} + m_2 \frac{\partial H_0}{\partial p_2} = 0 \tag{31}$$

is satisfied in every small region of (p_1, p_2). The vanishing of the Poisson bracket brings the functional dependency of H_0 and Φ_0 at infinite number of commensurable frequencies. This leads to the analytical functional dependency of Φ_0 and H_0, which contradicts the independency of Φ_0 and H_0.

Ever since the proofs of algebraic and analytic non-integrability of the three-body problem by Bruns and Poincaré that we have summarized, there might have been an acceptance that the study of integrable systems would not contribute to the understanding of general systems. This is because integrability requires a stringent condition, and also because integrable systems are special and very rare. General systems must be treated perturbatively or numerically. Of course, a deeper understanding of the structure of Hamiltonian or Lagrangian equations of motion or mechanical systems is required as well. Looking at the swarm of researchers attacking Hamiltonian systems from various aspects, it seems to us that the study of integrability is one of the most orthodox paths leading to the next reformulation or reinterpretation of Hamiltonian mechanics in general.

In the latter half of the twentieth century, the notion of integrability itself evolved. In the 1960s, geometrical interpretation was given to the concept of complete integrability. Naive integrability may just mean an integration of equations of motion by quadrature. Hamiltonian systems with n degrees of freedom are said to be integrable in the sense of Liouville, or simply Liouville integrable when there are n independent first integrals in involution. Here those first integrals are said to be in involution if Poisson brackets of each pair of them vanish. The first integral $I(\mathbf{x})$ is a

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lisional manifold using the McGehee variables. Readers can consult McGehee (1974) for the case of one-dimensional three-body problem, as well as Devaney (1980) for the case of planar isosceles three-body problem. Another, simpler example is a system described by a Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + q_1^2 q_2 + \varepsilon \frac{q_2^3}{3} \tag{27}$$

with a fixed perturbation parameter ε. A dynamical system dominated by H is algebraically integrable when $\varepsilon = 0, 1, 6$ or 16 (Yoshida, 1983a,b).

function whose time derivative in the direction of the vector field vanishes. Arnold added geometrical interpretation to this notion (Arnold, 1978). The Arnold–Liouville integrability (in the simplest setting) in addition to the statement of the Liouville integrability says that phase space is divided into n-dimensional invariant tori, and that the motion takes place in each of these tori.

According to Goriely (2000), there are three approaches to the study of integrability or non-integrability of dynamical systems: a dynamical system approach, an algebraic approach, and an analytic approach. A dynamical system approach starts with a search for fixed points or periodic points. This approach proceeds to an analysis of their stability, followed by various analyses such as topological conjugacy, bifurcation analysis, and so on. This approach seems to fit the study of non-integrable systems.

The algebraic approach has its origin in Bruns. One looks for the first integrals of algebraic functions, in particular, polynomial functions. If one substitutes a polynomial of degree k into the equation satisfied by the supposed first integral, then the problem of finding the first integral is reduced to an algebraic problem: the problem of solving a system of linear equations for its parameters (or coefficients). There is a notion of "second integral". Second integral is an invariant relation for a subset of phase space, and it serves as the building blocks for the first integral.

In the analytic approach, one looks at the local behavior of the solutions of equations of motion around their singularities in complex time. The singularity may be a pole, a branch point, or an essential singularity. The coefficient matrix of the variational equations around some particular solution is called Kovalevskaya matrix, and its eigenvalues are called Kovalevskaya exponents (Yoshida, 1983a). It has been shown that the behavior of these exponents is strongly related to the integrability of the system (Yoshida, 1983b). In this approach, one looks for meromorphic integrals, i.e. holomorphic (or analytic) integrals that do not have singularities other than poles.

The study of integrability was partly revived when solitons were found in partial differential equations. The Hirota method (Hirota and Satsuma, 1976) and the famous GGKM paper (Gardner *et al.*, 1967) should be consulted here. The study of integrability before 1983 was extensively summarized by Kozlov (1983).

In the mid 1970s, the integrability of the finite Toda lattice was proved (Hénon, 1974; Flaschka, 1974). Ziglin (1983) gave a necessary condition for the integrability of Hamiltonian systems using variational equations around particular solutions. This condition can be used as sufficient for the nonintegrability of the system. Yoshida (1983a,b) has made a pioneering contribution to the integrability and nonintegrability of Hamiltonian systems. Yoshida proposed a procedure to obtain necessary conditions for the weighted-homogeneous systems of differential equations to be integrable. Algebraic integrals do not exist if none of the Kovalevskaya exponents is rational, where Kovalevskaya exponents characterize the stability of certain singular solutions. This property of Kovalevskaya exponents is related to the single-valuedness of the solution of differential equations.

Yoshida considers a similarity invariant system of differential equations. Here, similarity invariance is the invariance under transformation

$$\begin{cases} t \to \alpha^{-1} t, \\ x_1 \to \alpha^{g_1} x_1, \\ x_2 \to \alpha^{g_2} x_2, \\ \quad \vdots \\ x_n \to \alpha^{g_n} x_n, \end{cases} \tag{32}$$

with rational numbers $g_1, \ldots, g_n$ and a constant α. This system of equations permits a particular solution originating in the similarity itself and corresponding to a collision (collapse) orbit in n-body systems. Solutions of the variational equations around this particular solution can be obtained from the eigenvalues and eigenvectors of an $n \times n$ complex constant matrix $K = (K_{ij})$ with

$$K_{ij} = \frac{\partial F_i}{\partial x_j}(c_1, \ldots, c_n) + \delta_{ij} g_i \tag{33}$$

formed from the coefficients of the variational equations. This matrix is given the name "Kovalevskaya" after the celebrated contribution of S. V. Kovalevskaya on the motion of a rigid body around a fixed point in the late nineteenth century. The characteristic equations for the Kovalevskaya matrix is called the "Kovalevskaya determinant", and the roots of this equation are called the "Kovalevskaya exponents". Later, these notions play important roles in the research in this area.

Now Yoshida introduces a weighted homogeneous polynomial ϕ of weighted degree M as

$$\phi(\alpha^{-1} t, \alpha^{g_1} x_1, \ldots, \alpha^{g_n} x_n) = \alpha^M \phi(t, x_1, \ldots, x_n). \tag{34}$$

Repeating Bruns' discussion on the non-existence of algebraic integrals, Yoshida reduces the existence of integrals into the existence of the rational first integrals of weighted homogeneous functions. Then he arrives at the main results. If the similarity invariant system of differential equations has a weighted homogeneous first integral of degree M, then M is a Kovalevskaya exponent. If the system is described by a Hamiltonian, and if the Hamiltonian has a weighted degree h and admits a weighted homogeneous first integral of weighted degree M, then M and $h - 1 - M$ are Kovalevskaya exponents.

Yoshida (1983b) shows that irrational or imaginary Kovalevskaya exponents are inconsistent with the existence of a sufficient number of algebraic first integrals. Before showing this, he introduces the algebraic integrability of general differential equations. Then he tries to extend his result to non-similarity invariant systems.

The integrability of Hamiltonian systems and the single-valuedness of solutions near singularities have long been suspected to have intimate connections. A system of ordinary differential equations is said to have the Painlevé property when its general solution does not have movable critical singularities. Here, a movable singularity means that the position of the singularity depends upon the initial conditions. A "critical singularity" means that a single-valued function, after analytic continuation, has different values at this singularity. The procedure for checking whether or not a given system has the Painlevé property is called the Painlevé test. Though it is widely believed that the Painlevé property is incompatible with chaotic motions, there is no rigorous proof of this simple statement yet to date (Goriely, 2000). For reference, Morales-Ruiz and Ramis (2001a,b) obtained a strong necessary condition for the integrability of Hamiltonian systems, extending the work of Ziglin (1983). The condition says that if the original Hamiltonian system is Liouville-integrable, then the variational equation around a particular solution is solvable by a combination of quadratures, exponential of quadratures, and algebraic functions.

Applying a theorem of Morales-Ruiz and Ramis (2001a,b) to the two-degrees-of-freedom Hamiltonian, Yoshida (1999) solved the weak Painlevé conjecture positively. The conjecture in Yoshida (1999) says that when the Hamiltonian $H = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(x_1, \ldots, x_n)$ has a weak Painlevé property, then it is Liouville integrable. As to the definition of the weak Painlevé property, see Goriely (2000) for example. Nakagawa and Yoshida (2001a,b) derived a necessary condition for the integrability of homogeneous Hamiltonian systems with two degrees of freedom. They made a list of all integrable two-dimensional homogeneous polynomial potentials with a polynomial integral of the order of at most four in momenta.

It has been shown that the transversal intersections of stable and unstable manifolds, i.e. the existence of a horseshoe, implies the non-integrability of dynamical systems (Moser, 1974). Conversely, the non-integrability implies the existence of transversal intersections of stable and unstable manifolds in some dynamical systems. One famous example is the standard map (Lazutkin *et al.*, 1989). It seems strange to the authors that the latter property is not yet proved rigorously. Numerically, the latter property seems obvious. In fact, "chaos comes from infinity," i.e. the number of iterates necessary for completing the horseshoe becomes infinite as the system approaches integrable.

Suppose we take a look at an arbitrary small neighborhood of a point in phase space. How are the set of solutions passing near the point geometrically disposed? The disposition is closely related to the integrability of the system. The range of potential dispositions increases with the degrees of freedom. At one extreme, there is a possibility that phase space is strictly stratified by hypersurface. At the other extreme, any solution curve wanders through all dimensions in the neighborhood of the point. This difference should be reflected in the functional form of solutions.

3. The three-body problem

The three-body problem has a long history, and there has been a great deal of previous research. This problem has attracted so many astronomers, physicists, and mathematicians, in particular in the twentieth century. This is because this problem contains many aspects of general dynamical systems. Let us point out some of them: The three-body problem contains (collisional) singularities which give rise to rich dynamical phenomena. There are various kinds of periodic orbits and related homoclinic and heteroclinic phenomena in the three-body problem. This problem can be used as a testbed of newly devised numerical methods such as symmetric schemes and symplectic integrators. New concept of integrability, such as integrability in the sense of C^n, is introduced as a part of the three-body problem research. There are various versions of the three-body problem: restricted, collinear, isosceles, planar, and three dimensional, each of which asserts its own *raison d'etre.* For a particular choice of mass, this problem is reduced to perturbed two-body problem that is close to be integrable. Thus, the three-body problem covers from near-integrable realm to chaotic realm of dynamical systems. The degree of freedom is from two to four. The three-body problem has many applications in astronomy and in physics. Triple interactions of particles are regarded as a basic process in the dynamics of stars and galaxies. Solutions like oscillatory and non-collision singularity, or "figure-eight", may be particular to this problem. Finally but not the last, the three-body problem is beautiful as it is.

In this section we categorize the study of the three-body problem into a few parts: Analytical and qualitative studies, numerical studies, and hierarchical triple systems. We also mention some current frontiers of three-body problem research including some recent applications to actual stellar and planetary systems. Due to our inability, many important topics related to the three-body problem are not very well described in this section, such as the restricted three-body problem as an independent field of study.

3.1. *Analytical and qualitative studies*

After the rigorous proofs of algebraic and analytical non-integrability by Bruns and Poincaré at the end of the nineteenth century (see Section 2.4), studies of the three-body problem moved toward qualitative directions (cf. Whittaker, 1904). Two famous example problems were proposed. One is about the existence of non-collision singularity in N-body systems proposed by Painlevé (1897). The other is about the existence of oscillatory solutions in the three-body system (Chazy, 1922). The former was solved by Xia (1992). The latter was first solved by Sitnikov (1960) for the restricted three-body problem, then later solved through a perturbative approach in the three-dimensional isosceles three-body problem by Alekseev (1968a,b, 1969), and finally, in the planar isosceles problem by Xia (1994). C. Siegel questioned why the three-body problem was not integrable. Siegel (1941) showed that triple collision is an essential singularity, and that triple collision is the origin of bad behaviors in the three-body problem.

Chazy (1922) described all possible types of final motion, i.e., behavior as $t \to \pm\infty$, of the three-body problem. After Chazy, from the 1940s to the 1960s, qualitative studies of the three-body problem flourished in the former Soviet Union (USSR). Final motions are classified into *H, P, E, B,* and *OS,* corresponding to hyperbolic, parabolic, elliptic, bounded, and oscillatory motion. The initial motion is suffixed by $-$ and final motions are suffixed by $+$. For example, $HE_j^- \to HE_i^+$ means that as $t \to -\infty$, one particle escapes (H) and the remaining two particles form a binary (E_j) where the escaping particle is the j–th particle. As $t \to \infty$, one particle escapes (H) and the remaining two particles form a binary (E_i) where the escaping particle is the i–th particle. In this particular example, the motion is called "exchange" when $i \neq j$, and the measure of this kind of orbits in phase space has been proved to be positive.

Classifications of the final and initial motions of the three-body problem were intensively carried out (e.g. Merman, 1958; Khilmi, 1961). This was a modest approach compared with the conventional pursuit of individual orbits. In the new approach, we are content with understanding the behavior of a bundle of orbits. The possibility of capture, escape, and exchange was discussed, and some important quantitative results have been obtained. See the reviews by Hagihara (1971) and Alekseev (1981) for more detail.

Saari (1971) studied the relation between collision singularity and the N-body problem. McGehee (1974) introduced a remarkable set of coordinates in the three-body problem. The coordinates are remarkable in the sense that triple collision is blown up to a manifold and orbits approaching triple collision move along the fictitious orbits on this collision manifold. Owing to these coordinates, theoretical research have been revitalized. People such as R. Moeckel, C. Simo, and R. L. Devaney have deployed the McGehee variables and analyzed the behavior of solutions of the three-body problem near triple collision. Recently, computer-aided research has tried to reveal the structure of general phase space in various settings (e.g. Tanikawa *et al.*, 1995; Umehara, 1997; Zare and Chesley, 1998; Umehara and Tanikawa, 2000; Tanikawa and Mikkola, 2000a,b; Tanikawa, 2000). There is a special field of study in which escape criteria are the target (e.g. Marshal, 1990); Yoshida's criteria (Yoshida, 1972, 1974) are considered as the best at present.

3.2. *Numerical studies*

Numerical studies of the three-body problem have a relatively long history, dating back to an era before the arrival of digital computers. Numerical integrations of the periodic solutions of the restricted three-body problem were carried out by two groups in the first half of the twentieth century, either by hand or using the Tiger accumulators. One of these groups was called the Copenhagen school, headed by E. Strömgren (Strömgren, 1935). The other was a Japanese group headed by T. Matukuma (Matukuma, 1933). After the introduction of electronic computers, Hénon (1965) systematically followed the scenario of the Copenhagen school, and started the quest for periodic orbits. Throughout the latter half of the twentieth century, following the introduction of electronic computers, the restricted three-body problem has been used as a mathematical model to express real dynamical phenomena in the real world. Examples are the studies on instability of asteroidal motion (e.g. Wisdom, 1983), origin of planetary spins (e.g. Tanikawa *et al.*, 1989, 1991), collision probability of planetesimals (e.g. Nakazawa *et al.*, 1989a,b), stability of planetary ring systems (e.g. Hénon and Petit, 1986; Petit and Hénon, 1987), or impossibility of the capture of satellites (e.g. Tanikawa, 1983).

Hénon (1965) was one of the first group of people to have started the numerical check of the KAM theory and the numerical study of the stability of periodic orbits, using the restricted three-body problem as a tool. It seems obvious that Hénon followed the tradition of Poincaré, treating the restricted three-body problem as a representative example of dynamical systems. In addition, Hénon was among the first generation of so-called "hackers". Since Hénon, the restricted three-body problem has been the target of research, as well as a good testbed, for electronic computers. V. Szebehely, W. H. Jefferys, V. V. Markellos, and many other people joined these researches at the frontier.

In addition to the numerical study of the restricted three-body problem, numerical integrations of the general three-body problem also started along with the popularization of electronic computers. Celestial mechanists all over the world began to take part, such as S. Aarseth (UK), J. P. Anosova (USSR), R. Broucke (USA), S. Mikkola (Finland), D. C. Heggie and P. Hut (UK), K. Zare (USA), V. Szebehely (USA), G. Contopoulos (Greece), S. Yabushita (Japan), or K. Tanikawa (Japan). There are a lot of variations within the general three-body problem: the collinear problem, planar and spatial isosceles problem, general planar problem, hierarchical problem, free-fall problem, and many others. For astronomical application, encounters of single and binary stars are important, and the timescale for the disintegration of such systems has been studied (e.g. Heggie, 1975; Heggie and Hut, 1993). The research on stability of the hierarchical three-body problem that was started by Harrington (1975) may be closely related to the study of extrasolar planetary systems.

Numerical study of the three-body problems have also demonstrated how difficult accurate numerical integrations of gravitational N-body problems are, even when N is not large. For example, Szebehely and Peters (1967) numerically solved the three-body Pythagorean problem in which the three bodies initially fixed on the vertices of a Pythagorean triangle with edges of lengths 3, 4, and 5, providing a movie of the results as well. People looking at this movie understood the difficulty in predicting final motions of the three-body problem with many close encounters. Miller (1964) numerically integrated back and fourth a few-body problem and showed the deterioration of the accuracy when the system experiences a close encounter. He measured the accuracy by the phase distance of the system at $t = 0$ and the system returned to $t = 0$ after some excursions in the phase space. Miller pointed out that Hamiltonian systems are exponentially unstable, and that this causes unreliability of numerical integrations. At nearly the same time, international comparison of the accuracy of numerical integrations using the same 25-body problem has been carried out (Lecar, 1968; Hayli, 1970). Nine of eleven participating groups integrated "the standard" 25-body problem beginning with the free-fall initial conditions until $\sim$ 2.5 crossing times. It has been shown that depending on computers, different bodies escape from the system at different times. Further, the number of escapers differs from computer to computer. The problem raised in this comparison is still open. Another comparison using 32 bodies was performed (Miller, 1971a,b), which raised a similar question (Aarseth and Lecar, 1975). These difficulties in numerical integration of N-body systems have led us to the development of more sophisticated numerical techniques, especially those of regularization (see Section 5.1).

3.3. *Hierarchical triple systems*

It might not be wrong to say that most of the three-body systems in the universe have hierarchical structures with a high contrast of component masses. In these systems, declaring a clear definition or a clear criterion of stability and instability is essential.

So far there are three different stability criteria in the literature relevant to hierarchical triple systems. First: a hierarchical system is stable if there is no escape of a body (e.g. Anosova, 1996). The stability in this sense is called the *E*-stability. Second: a hierarchical system is stable if there is no change in configuration (e.g. Szebehely and Zare, 1977; Kiseleva *et al.*, 1994a,b). Here the configuration is defined as being changed when the partner of the inner binary is replaced, or when the hierarchical configuration itself is lost. This stability is called the *CN*-stability. Third: a hierarchical system is stable if there are neither secular changes nor large variations in semimajor axis and/or eccentricity (e.g. Harrington, 1975; Black, 1982; Donnison and Mikulskis, 1992). The stability in this sense is called the *T*-stability.

Among the three stability criteria, the *T*-stability is, as Harrington himself admitted, subjective and rather a poorly specifying one. For example, Black (1982) considered that the upper limit of the variations of semimajor axis should be 10%. He thought that systems with larger variations would be *E*-unstable. The value of 10% has no theoretical background. Hence the criterion may change from author to author, as Black (1982) or Pendleton and Black (1983) indicated.

The *CN*-stability is a better-defined concept since two-body energies of pairs can be estimated from the orbital information. However, the change of the three-body configuration that this criterion supposes might be slightly too drastic, judging from that fact that there exist triple stellar systems such as the CH Cyg system (e.g. Mikkola and Tanikawa, 1998a) that show cataclysmic variation presumably without a drastic change in three-body configuration. We need another, milder criterion.

Here we propose yet another stability criterion: A hierarchical system is stable when there occurs neither collision of components nor escape of a body. Hereafter we call the stability in this sense as *C*-stability. Suppose that a collision takes place between the inner binary components. Then, the *C*-stability criterion distinguishes prograde and retrograde hierarchical systems.

Checking the *C*-stability numerically is easy, if we give finite non-zero radii to the components, or if we use a numerical technique developed by Tanikawa *et al.* (1995). It is conceivable that near-collision orbits develop changes in configuration. Therefore, we cannot say in general which of *CN*- and *C*-stabilities is stronger.

To organize the relationship between *E*-, *C*-, and *T*-stabilities, consider the sets of *E*-, *C*-, and *T*-stable trajectories in phase space for a given combination of masses. Evidently, we have $T \subset C \subset E$. T is a proper subset of C. On the other hand, C and E may have common boundaries, i.e., some *C*-unstable systems may also be *E*-unstable.

There is a series of numerical studies for the stability of hierarchical triples initiated by Harrington (1975, 1977). His criterion is that there is no secular change in a and e of the triple. Let q_2 be the periastron of the outer binary and a_1 be the semimajor axis of the inner binary. Three masses are m_1, m_2, and m_3, among which m_3 is of the outer body. Harrington proposed a sufficient condition for stability of the planar three-body problem in the form

$$\frac{q_2}{a_1} \geq \frac{(q_2/a_1)_0}{\log_{10} 1.5} \log_{10}\left(1 + \frac{m_3}{m_1 + m_2}\right), \tag{35}$$

where $(q_2/a_1)_0$ is the least stable value for the equal-mass triples. $(q_2/a_1)_0$ has different values depending on the direction of rotation of the outer body. Following Harrington, Szebehely and Zare (1977), Black (1982), Donnison and Mikulskis (1992) derived better criteria. The newest criterion is due to Mardling and Aarseth (1999). Their criterion for the planar prograde motion is

$$\frac{q_2}{a_1} \geq C\left[\left(1+\frac{m_3}{m_1+m_2}\right)\frac{1+e_{\rm out}}{\sqrt{1-e_{\rm out}}}\right]^{\frac{2}{5}}, \tag{36}$$

where $e_{\rm out}$ is the eccentricity of the outer binary. The constant $C \simeq 2.8$ is determined empirically.

As we will mention later (such as in Section 4.2), recent investigations into the long-term stability of planetary orbits have exhibited a large volume of chaotic orbits that are actually stable in the sense that they survive billions of years without resulting in collisions between planets. These results suggest the existence a class of orbits that are chaotic as well as C-stable in N-body systems with $N > 3$.

3.4. *New frontiers of the three-body problems*

The three-body problem has long been an attractive but dangerous subject for students. This is because the three-body problem has quite a simple setting that even a high school student can understand, and it appears relatively easy to find something new even for a beginner. However, the three-body problem has been investigated in such a great detail for so many years that it is actually very difficult for a newcomer to quickly obtain anything new in this field. Thus, experienced scientists often give an alert to new students, "Never be attracted to the three-body problem. It is too dangerous."

Although purely theoretical aspects of the three-body problem are quite formidable for students, the advent of fast computers allows us to observe and analyze the ever-increasing complexity of the phase space created by the three-body problem. There is an opinion that the general three-body problem can only survive as a toy for eccentric scientists. What is the purpose of solving the motion only of three particles? Is there any relation to the other field of sciences even if one knows how three bodies move? Fortunately, however, the three-body problem has rather strong connections to other fields. From a practical point of view, the three-body problem has yielded many new applications to astrophysics. Some examples are given in Section 3.5. From a theoretical point of view, connections between the three-body problem and general Hamiltonian systems are rather strong. Indeed, the equations of motion of the three-body problem are among the simplest and the most meaningful examples of differential equations appearing in astronomy, physics, and mathematics. Moreover, the equations are non-integrable: the set of solutions may have a very complicated and intriguing structure. Theoretical and numerical techniques developed in general Hamiltonian systems and in general differential equations have been checked of their effectiveness in terms of the three-body problem. Conversely, techniques developed in the three-body problem have been extended to a broader class of differential equations. This tendency will continue in the new century.

One promising approach to the three- and several-body problem is to apply symbolic dynamics. It might be called "numerical symbolic dynamics". In symbolic dynamics, an orbit is replaced by a finite, infinite, or bi-infinite sequence of symbols that represent special events along the orbit. Typically, a special event corresponds to the occupation of special place in phase space. The number of special places is equal to the number of symbols necessary to construct a symbol sequence. The origin of symbolic dynamics dates back to Hadamard (1898) and the textbook of Birkhoff (1927); the method was soon developed in a paper entitled *Symbolic Dynamics* by Morse and Hedlund (1938). Alekseev (1968a,b) applied symbolic dynamics to the three-body problem (see also Alekseev, 1981). Alekseev used symbolic dynamics to prove the existence of oscillatory orbits in the general three-body system. On an oscillatory orbit of the three-body system, one of the bodies is repeatedly ejected from the remaining two bodies, and the amplitude of the ejection distance becomes unbounded in time.

Numerical symbolic dynamics has recently been used to unveil qualitative properties of three-body orbits. Examples of successful applications of numerical symbolic dynamics to the few-body problem are Tanikawa and Mikkola (2000a,b), Zare and Chesley (1998), Sekiguchi and Tanikawa (2004), Sano (2004), and Saito (2005). Tanikawa and Mikkola (2000a,b), in the one-dimensional three-body problem, explicitly used two types of binary collision and a triple collision along orbits as three symbols, and replaced the orbits with sequences of these three symbols. Using general techniques of symbolic dynamics such as making the transition graph, and searching inadmissible symbol sequences and periodic sequences, they succeeded in obtaining qualitative properties of orbits and phase space structure of the one-dimensional three-body problem with equal mass. These studies happened to clarify the relation between the phase space structure and triple collisions. Collision orbits are on the stable or unstable manifolds of the fixed points on the collision manifold in the extended phase space. It has been clearly illustrated that stable and unstable manifolds form the backbone of the phase space. In addition, triple collisions have been easily identified. This method has been applied to the symmetric four-body problem (Sekiguchi and Tanikawa, 2004) and the collinear Helium nucleus (eZe) problem (Sano, 2004). The technique is expected to be applicable to the planar three-body problem with zero angular momentum.

At the turn of the century, unexpected and surprising news spread through the celestial mechanics community: A proof of the existence of a new type of periodic orbit, the figure-eight ("8") orbit (Chenciner and Montgomery, 2000). This solution has been called "a choreographic solution" (Simó, 2002). This is because bodies dance along a single track on the plane in this type of orbits, with all the bodies having the same mass with time shift equal to 1/3 of the period. The figure-eight solution lives on the hypersurface of zero angular momentum, and it is stable. Chenciner and Montgomery (2000) noted that the space of oriented triangles in the plane formed by three bodies up to translation and rotation, if normalized by the inertial moment, is a 2-sphere. They called this the "shape sphere". The north and south poles correspond to equilateral triangles (Lagrange configurations), the equator corresponds to degenerate triangles (collinear configurations), and three equally separated meridians correspond to isosceles triangles. Cross points of these meridians and the equator are either collinear Euler configurations E_i, or binary collisions C_i. The shape sphere is divided into twelve similar regions by three meridians and the equator. A path in one of the twelve regions, starting at the Euler configuration and ending at an isosceles configuration, is copied to the remaining eleven regions with the aid of symmetry. Upon connection, these twelve paths form one closed path on the shape sphere. Chenciner and Montgomery (2000) first showed that the action-minimizing path has zero angular momentum. Then they showed that the action-minimizing path does not experience collision. Finally they showed that the connected path actually corresponds to a closed path in the configuration space and to the figure-eight solution.

Simó (2002) studied numerically the figure-eight orbit and its neighboring orbits. He took a surface of section near a phase space point where the figure-eight orbit becomes collinear. He integrated 36 million orbits around there until one of collision, long ejection to some distance, and the completion of 5,000 revolutions of the figure-eight orbit occurs. He plotted the initial points of the surviving orbits, and discussed various properties of these orbits. There was no choreographic orbit other than the figure-eight orbit on this local surface of section. Then Simo changed his strategy. He extended the figure-eight orbit problem in two different directions. One direction was to try different mass combinations. He found periodic orbits, but they were non-choreographic. The other direction was to give non-zero angular momentum. In this case, Simo found a lot of choreographic orbits: the figure-eight rotates and closes after some revolutions.

Seven years before Chenciner and Montgomery (2000), C. Moore (1993) had numerically found the figure-eight periodic solution of the three-body problem. His ideas are remarkable in three-fold aspects. First, Moore considered the planar N-body problem and construct braids in 2+1 space-time. Second, Moore embedded the gravitational N-body problem in the dynamics with potential

$$V = \sum_{i}^{N} \sum_{j}^{N} V_{ij}, \quad V_{ij} = A m_i m_j r_{ij}^{\alpha} \quad (-2 \leq \alpha \leq 2). \tag{37}$$

The third idea is that Moore put the problem in the extrema principle. He observed that the system is integrable for $\alpha = 2$, and that the bodies move around the center of mass harmonically. So, all braids are trivial when $\alpha = 2$, whereas all braid types exist when $\alpha = -2$. For intermediate values of α, some braids exist, while others do not: In the word of the N-body problem, some periodic orbits exist, while others do not. Moore confirmed that the changes in accordance with α is monotone. To obtain periodic orbits, Moore first specified the braid type, and started with a test orbit that realizes this braid type. His action reads

$$\partial_\tau S = -m \int \left(\frac{d^2\mathbf{x}}{dt^2} - \frac{F(\mathbf{x})}{m} \right)^2 dt. \tag{38}$$

This becomes maximal when the acceleration on the path is derived from force. Moore moved from one path to another in order that $\partial_\tau S$ increases. If a collision takes place before the action maximizing path is attained, then there is no periodic orbit corresponding to the given braid, because the braid type changes through collision. In this way, Moore found the figure-eight orbit for $\alpha < 2$. He found other kind of orbits as well. However, the figure-eight orbit is the only choreographic orbit (see Table I of Moore (1993)). We might want to look for other choreographic orbits using Moore's method.

The discovery of the figure-eight orbit triggered an explosion of the study of choreographic (or dancing) N-body solutions (note that these names were given by C. Simó). Together with many other publications, works by T. Fujiwara and collaborators are worth being mentioned (Fujiwara *et al.*, 2003a,b, 2004a; Fujiwara and Montgomery, 2003). They found very interesting and simple properties of the zero angular momentum solution of the three-body problem. One is the three-tangents theorem which says that in the center of gravity system, three tangents to the directions of motion of mass points always cross at a point or at infinity. Similarly, in the center of gravity system, if the moment of inertia is constant, then normal lines to the velocity vectors at bodies meet at a point or at infinity. Kuwabara and Tanikawa (2006) extended the three-tangents theorem and three-normals theorem to the planar three-body problem with non-zero angular momentum. Fujiwara and his collaborators also showed that the form of the figure-eight solution is a lemniscate when the force has an additional repulsive term (Fujiwara *et al.*, 2003a). One other property they have shown is the convexity of the figure-eight orbit (Fujiwara *et al.*, 2003b). Though the exact functional form of the figure-eight orbit is still unknown, the figure-eight orbit turned out to have a non-constant moment of inertia (Fujiwara and Montgomery, 2003). Fujiwara *et al.* (2004a) suggested a direction of general study of the planar three-body problem with the aid of a three-tangents theorem.

Choreographic N-body solutions and related topics will remain an important and exciting subject in the early twenty-first century. A lot of solutions for the N-body problem of this kind have been found, and continuously being found. Mathematical structures of the solutions and their positions in phase space are attracting many scientists, which may affect all other branches of mathematical sciences as the three-body problem has done. It has been found that embedding the gravitational N-body problem in a sequence of different force laws characterizes the gravitational N-body problem itself. Works by Fujiwara and Moore are good examples. We will find further interesting issues of the choreographic N-body problem in this direction.

3.5. *Applications to actual planetary and stellar systems*

The three-body problem shows even larger importance when it is applied to actual planetary or stellar systems. Since there have been so many applications of the restricted three-body problem to astronomy, it is impossible to cover the whole range of applications in this manuscript. Some of them are dispersed in various sections. As for the future direction of the application, the restricted three-body problem will be applied to newly discovered multiple stellar systems, such as extrasolar planetary systems or stars around binary blackholes. As for the general three-body problem, the authors would like to point out two promising directions of study. The Kozai mechanism is involved in both of them.

The first one is the study of long-periodic variable stars, for example, a very strange variable, CH Cyg (Mikolajewski *et al.*, 1990). In the nineteenth century, CH Cyg used to be a standard star. Later it became a semi-regular star. Then, it suddenly started its prominent activity in 1963. Since then it has been almost constantly emitting radio, optical, and even X-ray jets. Its brightness changes by a few magnitude. Actually CH Cyg was recognized as a binary star (Yamashita and Maehara, 1979), but this kind of sudden start of stellar activity cannot be simply explained by a binary star model. Through infrared spectral observations, CH Cyg turned out to be a triple (Hinkle *et al.*, 1993). As a binary, it has a period of around two years. But as a triple, it has a period of 14 years. How can a triple like this maintain this long-periodic activity, apparently longer than a hundred years? The first trial to solve this problem was done by Mikkola and Tanikawa (1998b). They proposed the Kozai mechanism as a dynamical model to explain the behavior of CH Cyg. The Kozai mechanism has increasing importance in the various fields of astronomy. In the hierarchical three-body problem, the Kozai mechanism has a long periodicity called the Kozai cycle. The period T_{Kozai} of the Kozai cycle is numerically fitted to a formula as follows (Aarseth, 2003):

$$T_{\text{Kozai}} = \frac{T_{\text{out}}^2}{T_{\text{in}}}\left(\frac{1+q_{\text{out}}}{q_{\text{out}}}\right)\left(1-e_{\text{out}}^2\right)^{\frac{3}{2}} g(e_{\text{in}}, \omega_{\text{in}}, \psi) \qquad (39)$$

where T_{out} and e_{out} are the orbital period and eccentricity of the outer binary, T_{in}, e_{in}, ω_{in} are the orbital period, eccentricity, and argument of pericenter of the inner binary, ψ is the orbital inclination, $q_{\text{out}} = m_3/(m_1+m_2)$, and g is a function of order unity depending on its arguments. As is seen in (39), T_{Kozai} is longer than the orbital period of the outer component of a triple by a factor of ten or more. When the orbital planes of the three bodies coincide, the eccentricity of the inner binary increases. Hence at the pericenter, the supergiant component of CH Cyg may fill its Roche lobe. Overflowed material accretes onto a companion—neutron star or white dwarf. During this stage, CH Cyg repeats activity in two year periodicity. On the other hand when the orbital planes of two binaries differ, the inner binary becomes a detached system without any activity. This is the basic explanation for the strange behavior of CH Cyg.

Related to this study is the direct search for long periodic variables. For example, Fujiwara *et al.* (2004b) compared seven historical star catalogues from Almagest to Uranometria, and tried to find long periodic variables. Fujiwara and her colleagues found 57 Peg and 19 Psc as candidates (Fujiwara and Hirai, 2006). Dynamical work (Mikkola and Tanikawa, 1998b) and astrophysical work (Fujiwara and Hirai, 2006) may give impetus to the study of variable stars that have been considered irregular until now and left unanalyzed. In this respect, a systematic dynamical study of three-dimensional hierarchical triple systems is strongly desired.

The second example of the promising directions of the general three-body problem is the study of blackholes in the center of galaxies which recently draw a large attention. Merged galaxies may contain two or more super-massive blackholes in their center. Then, for the evolution of the entire galaxy, the evolution of central super-massive blackholes is crucial. There arises a relativistic three-body problem. If mergings are numerous, the frequency of triple blackholes is high. There can be a considerable change in the fate of the central part of a galaxy whether it has a black hole binary or a triple blackhole. Merging timescale of a binary blackhole is said to be longer than the age of the universe. Iwasawa *et al.* (2006) studied this problem. They found that three-dimensional three-body problem of supermassive blackholes significantly shorten the timescale. Indeed, as in the case of stellar triples, there are configurations in which the Kozai mechanism is effective. In these cases, when the orbital planes coincide, the inner binary may emit gravitational waves at their pericenter and their orbit shrinks. This substantially speeds up the orbital evolution of the inner binary. Then, the binary merges during the timescale shorter than the Hubble time.

4. Solar system dynamics

In the latter half of the nineteenth century, discoveries of major and minor planets were the most significant issue in celestial mechanics. Also, earlier in the twentieth century, the existence of another planet was predicted, and the effort to find it bore fruit when C. W. Tombaugh made his discovery of Pluto in 1930. This kind of activity was directly connected to the development of analytical perturbation theories, now gradually being replaced by numerical integrations. In what follows, we describe several major aspects of the research in solar system dynamics: progress of analytical perturbation theories, long-term dynamical stability of the solar system planetary motion, dynamical studies of planet formation, dynamical studies of small bodies, and planetary rotation. Many important topics related to solar system dynamics are not described here due to authors' incompetence such as dynamics of near-Earth asteroids and that of ring-satellite systems, motion and control of artificial satellites and spacecrafts, theory of planetary tides, interaction between solar system bodies and protoplanetary nebula, and orbital determination. The description about bodies at the outer edge of the solar system (trans-Neptunian objects including the Kuiper Belt, the scattered disk, and the Oort Cloud objects) is extremely limited, and should be expanded far more. The subsection 4.5 for the dynamics of planetary rotation should have been longer and more extensive, incorporating the latest observation results and their dynamical interpretation. Also, we do not have a specific section about extrasolar planetary systems that offer us an extremely large variety of testbeds for solar system dynamics. This is because we think the most methodology of extrasolar planetary systems research is already embedded in the conventional studies of solar system bodies, many of which we have described in this manuscript.

4.1. *Progress of perturbation theories*

In the era of Laplace or Lagrange, solar system planetary motion and its stability were the central issues of astronomy. The perturbation equations of orbital elements derived by the method of variation of constant are now called Lagrange's planetary equations. Also, a secular perturbation theory up to the lowest order in planetary eccentricity and inclination is called the Laplace–Lagrange method. In the nineteenth century, urged on by the advent of accuracy in astronomical observation of planetary motion, greater precision and reliability were required of analytical perturbation theories in order to describe and predict planetary motion. Many famous astronomers such as U. J. J. LeVerrier, S. Newcomb, and G. W. Hill devoted a large part of their lives to construct accurate perturbation solutions of planetary motion.

In addition to planetary motion, lunar motion had drawn the attention of scientists working on analytical perturbation theories until the end of the nineteenth century. Newton showed in *Principia* that the principal periodic perturbation in lunar motion can be ascribed to the action of the Sun. For constructing the accurate ephemeris of the Moon, the method of the variation of arbitrary constants was studied in the eighteenth century by Euler, and was later discussed in more detail by Poisson in the mid nineteenth century. This method, which had been an important foundation of canonical perturbation theories in the twentieth century, was further developed by Delaunay in the 1860s. Delaunay completed the application of his method to the solution of the lunar "main problem" where only the Moon, the Earth, and the Sun are taken into account and considered as point masses. After Delaunay in the 1880s, Hill provided the calculation of the terms arising from the non-sphericity of the Earth, and Radau calculated the planetary perturbations in the 1890s. All this work created a strong basis for the analytical perturbation theories that were developed later in the twentieth century. Brouwer and Hori (1962) provide a concise summary of the history of lunar perturbation theories.

Based on the legacy of the nineteenth century, a classical planetary perturbation theory by Brouwer and van Woerkom (1950) was developed as an extension and a rebuild of Hill's theory (Hill, 1897). Brouwer and van Woerkom (1950) included the terms up to the sixth degree in the eccentricities and inclinations of Jupiter and Saturn. They also considered terms related to the "great inequality" (i.e. the near equality to the 5:2 mean motion resonance) between Jupiter and Saturn up to the second order in planetary masses.

Bretagnon (1974) and Bretagnon and Francou (1992) extended the classical theory of Brouwer and van Woerkom (1950). Bretagnon's analytical theory includes the terms up to the fourth degree in planetary eccentricities and inclinations, and up to second order in planetary masses. Based on the preceding research, Laskar developed a very high-accuracy semi-analytical secular perturbation theory (Laskar, 1985, 1986, 1988, 1990). Laskar's theory includes terms up to the sixth degree in planetary eccentricities and inclinations, and up to second order in planetary masses. Laskar's secular perturbation theory is an extension of Duriez's (1977; 1982) preceding work concerning the four jovian planets. Laskar solved the equations of motion of secular planetary perturbation, decomposed the solutions into Fourier components, and obtained the frequencies and the amplitudes of major periodic oscillations. This theory now serves as a standard in the area of classical secular planetary theories.

After the epochal launch of Sputnik in 1957, the application of analytical perturbation theories suddenly expanded toward the motion of artificial satellites. Starting from the three famous papers (Garfinkel, 1959; Kozai, 1959; Brouwer, 1959) published consecutively in the same issue of the same journal, a large number of theories have been published about the motion of artificial satellites around the Earth, composing the fundamental theoretical basis of the more than 5,000 artificial satellites that have been launched since Sputnik. Kozai (1962a) and Kinoshita (1977b) established very high-accuracy theories of the motion of artificial satellites. Analytical perturbation theories for high-eccentricity and high-inclination satellite orbits have been developed by E. Brumberg and V. A. Brumberg, exploiting elliptic functions (Brumberg *et al.*, 1995; Brumberg and Brumberg, 1995, 2001). New theories are still being published in the field of satellite control and astronautics, with plenty of good and detailed textbooks dealing with this issue such as those by Taff (1985), Battin (1987), or Montenbruck and Gill (2000).

One class of the newer perturbation methods is based on Hamiltonian dynamics: The canonical perturbation theory. In a general problem with small perturbation, suppose that a simple, time-independent Hamiltonian H with n degrees of freedom be written as

$$\begin{aligned} &H(\theta_1,\theta_2,\cdots,\theta_n,J_1,J_2,\cdots,J_n)\\ &\quad = H_0(J_1,J_2,\cdots,J_n)\\ &\quad\quad + H_{\mathrm{pert}}(\theta_1,\theta_2,\cdots,\theta_n,J_1,J_2,\cdots,J_n) \end{aligned} \tag{40}$$

with Hamilton's canonical equations

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial J_i}, \quad \frac{dJ_i}{dt} = -\frac{\partial H}{\partial \theta_i} \quad (i = 1,\cdots,n) \tag{41}$$

where θ_i and J_i are the angles and actions of the unperturbed (and integrable) system, H_0. The perturbed part of the Hamiltonian H_{pert} is assumed to be much smaller than the unperturbed part (i.e. $H_{\mathrm{pert}}/H_0 = O(\varepsilon)$).

In canonical perturbation theories, we apply canonical transformations to the system described by (40) and (41), trying to reduce a system with terms of non-integrable nature (H_{pert}) to an integrable form. This procedure is equivalent to eliminating short-periodic terms (i.e. angle-like variables) from the Hamiltonian by averaging processes. Our eventual goal is to make the system depend only on actions which are constant. More specifically, we want to convert the Hamiltonian (40) by a canonical transformation into a form as

$$\begin{aligned} &H^*(-,\theta_2^*,\cdots,\theta_n^*,J_1^*,J_2^*,\cdots,J_n^*)\\ &\quad = H_0^*(J_1^*,J_2^*,\cdots,J_n^*)\\ &\quad\quad + H_{\mathrm{pert}}^*(-,\theta_2^*,\cdots,\theta_n^*,J_1^*,J_2^*,\cdots,J_n^*) \end{aligned} \tag{42}$$

where "$-$" denotes that a variable is eliminated by an averaging process (in this case, θ_1 is supposed to be eliminated from the system), and the superscript $*$ denotes that this system has been canonically transformed. Canonical equations using a time-like variable t^* will be

$$\frac{d\theta_i^*}{dt^*} = \frac{\partial H^*}{\partial J_i^*}, \quad \frac{dJ_i^*}{dt^*} = -\frac{\partial H^*}{\partial \theta_i^*}, \tag{43}$$

for $i = 1,\cdots,n$. Similarly, we eliminate θ_2^* from H^* by another averaging process as

$$\begin{aligned} &H^{**}(-,-,\theta_3^{**},\cdots,\theta_n^{**},J_1^{**},J_2^{**},\cdots,J_n^{**})\\ &\quad = H_0^{**}(J_1^{**},J_2^{**},\cdots,J_n^{**})\\ &\quad\quad + H_{\mathrm{pert}}^{**}(-,-,\theta_3^{**},\cdots,\theta_n^{**},J_1^{**},J_2^{**},\cdots,J_n^{**}) \end{aligned} \tag{44}$$

with

$$\frac{d\theta_i^{**}}{dt^{**}} = \frac{\partial H^{**}}{\partial J_i^{**}}, \quad \frac{dJ_i^{**}}{dt^{**}} = -\frac{\partial H^{**}}{\partial \theta_i^{**}}, \tag{45}$$

for $i = 1,\cdots,n$. If this series of canonical transformations remains possible until we have eliminated all the angle-like variables, we will end up with obtaining a Hamiltonian that contains only actions as

$$\begin{aligned} &H^{*\cdots*}(-,\cdots,-,J_1^{*\cdots*},J_2^{*\cdots*},\cdots,J_n^{*\cdots*})\\ &\quad = H_0^{*\cdots*}(J_1^{*\cdots*},J_2^{*\cdots*},\cdots,J_n^{*\cdots*})\\ &\quad\quad + H_{\mathrm{pert}}^{*\cdots*}(-,\cdots,-,J_1^{*\cdots*},J_2^{*\cdots*},\cdots,J_n^{*\cdots*}) \end{aligned} \tag{46}$$

with a set of canonical equations of motion

$$\frac{d\theta_i^{*\cdots*}}{dt^{*\cdots*}} = \frac{\partial H^{*\cdots*}}{\partial J_i^{*\cdots*}}, \quad \frac{dJ_i^{*\cdots*}}{dt^{*\cdots*}} = -\frac{\partial H^{*\cdots*}}{\partial \theta_i^{*\cdots*}}, \tag{47}$$

for $i = 1,\cdots,n$. Then, the system described by (40) and (41) would be integrable: The second equation of (47) tells us that all $J_i^{*\cdots*}$ are constant; they are true actions of $H^{*\cdots*}$. Also, from the first equation of (47), it turns out that the angles $\theta_i^{*\cdots*}$ are just proportional to the time-like variable $t^{*\cdots*}$.

In the field of general canonical perturbation methods, we should mention the epochmaking transformation theory of G. Hori (Hori, 1966, 1967, 1970, 1971) which utilizes the Lie transformation and accomplishes an explicit canonical transformation. This explicitness is the most significant advantage of Hori's method in its practical use, making it far more advanced than the traditional von Zeipel method (cf. von Zeipel, 1916). Later, Deprit (1969) proposed another form of the explicit canonical transformation theory equivalent to Hori's.

The explicitness of Hori's general perturbation theory may be summarized as follows (Hori, 1966). First, let us assume that (ξ_j,η_j) is a set of $2n$ canonical variables with $j = 1,2,\cdots,n$. Let $f(\xi_1,\xi_2,\cdots,\xi_n,\eta_1,\eta_2,\cdots,\eta_n)$ and $S(\xi_1,\xi_2,\cdots,\xi_n,\eta_1,\eta_2,\cdots,\eta_n)$ be arbitrary functions of (ξ_j,η_j). Here we introduce a differential operator D_S^n as:

$$\begin{aligned} &D_S^0 f = f,\\ &D_S^1 f = \{f, S\},\\ &D_S^n f = D_S^{n-1}(D_S^1 f), \quad (n \geq 2) \end{aligned} \tag{48}$$

with the Poisson brackets $\{,\}$. The theoretical basis that Hori (1966) relies upon is the fact that was discovered by Lie (1888): A set of $2n$ variables x_j, y_j $(j = 1,\cdots,n)$ defined by

$$\begin{aligned} &f(x_1,\cdots,x_n,y_1,\cdots,y_n)\\ &\quad = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} D_S^n f(\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n), \end{aligned} \tag{49}$$

is canonical if the series in the right-hand side of (49) converges, where ε is a small constant that is independent of ξ_j and η_j. When $f(\xi_1,\cdots,\eta_n) = \xi_j$ or $f(\xi_1,\cdots,\eta_n) = \eta_j$, from (49) we get

$$\begin{aligned} x_j &= \xi_j + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} D_S^{n-1} \frac{\partial S}{\partial \eta_j},\\ y_j &= \eta_j - \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} D_S^{n-1} \frac{\partial S}{\partial \xi_j}. \end{aligned} \tag{50}$$

What is most remarkable in (50) is that the new variables (x_j, y_j) and the original variables (ξ_j, η_j) are not mixed: x_j and y_j are only on the left-hand sides, while the right-hand sides of (50) are the functions only of ξ_j and η_j. We can compare (50) with the conventional form of canonical transformation as

$$x_j = \xi_j + \varepsilon \frac{\partial \tilde{S}}{\partial y_j}, \quad y_j = \eta_j - \varepsilon \frac{\partial \tilde{S}}{\partial \xi_j}, \tag{51}$$

where $\tilde{S}(\xi_1, \cdots, \xi_n, y_1, \cdots, y_n)$ is a function of the mixed set of original and new variables, ξ_i and y_i. When ε is small enough, $S(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n)$ and $\tilde{S}(\xi_1, \cdots, \xi_n, y_1, \cdots, y_n)$ are related to each other by the following equation:

$$\begin{aligned} S = \tilde{S} &- \frac{\varepsilon}{2} \sum_{j=1}^{n} \frac{\partial \tilde{S}}{\partial \xi_j} \frac{\partial \tilde{S}}{\partial \eta_j} \\ &+ \frac{\varepsilon^2}{12} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \tilde{S}}{\partial \xi_j} \frac{\partial \tilde{S}}{\partial \xi_k} \frac{\partial^2 \tilde{S}}{\partial \eta_j \partial \eta_k} \right. \\ &\quad \left. + 4 \frac{\partial \tilde{S}}{\partial \xi_j} \frac{\partial \tilde{S}}{\partial \eta_k} \frac{\partial^2 \tilde{S}}{\partial \eta_j \partial \xi_k} + \frac{\partial \tilde{S}}{\partial \eta_j} \frac{\partial \tilde{S}}{\partial \eta_k} \frac{\partial^2 \tilde{S}}{\partial \xi_j \partial \xi_k} \right) \\ &\quad + O(\varepsilon^3). \end{aligned} \tag{52}$$

In (51), the relationship between the new variables (x_j, y_j) and the original variables (ξ_j, η_j) is implicit, while it is explicit in (50). Obviously, the explicit relationship (50) is far more useful in many of the realistic problems of actual dynamical application of canonical perturbation theories.

One of the major obstacles for traditional analytical perturbation theories is that they are hard to use for treating the orbits that cross each other, such as those of near-Earth asteroids. This kind of orbit introduces a singularity in the Hamiltonian. The averaging of the equations of motion that is traditionally used to compute secular perturbations is not definable. This singularity problem has recently been overcome in a series of efforts in a secular theory by G. F. Gronchi (Gronchi and Milani, 1999a,b, 2001; Gronchi and Michel, 2001; Gronchi, 2002). He showed that it is possible to define generalized averaged equations of motion for crossed orbits. These are obtained using Kantorovich's method of extraction of singularities. The modified distance used to approximate the singularity is the same one used by Wetherill (1967).

Current perturbation theories, though they are still called "analytical," depend so strongly on computer algebra that we are better off calling them "computationally-analytical". Indeed in the near future, we may be able to anticipate the development of a "fully automatic analytical development software" package, to be used as a development tool exclusively for perturbation theories in celestial mechanics. When using this kind of software, all we have to do would be to specify orders and parameters for the development of perturbation. Then, computers would provide us with solutions as accurate as we desired. However, even if this ultimate situation arises, someone will have to understand the meaning of the output from the package, lest the package be a total black box. Thus, analytical perturbation theories should be passed on from one to another as an important traditional art in solar system dynamics. It might not be necessary that a huge number of people are involved, but there must be a certain number of people (or groups) that dedicate themselves to this kind of analytical work. We must also keep in mind that work in this field generally requires a much longer time to publish than does work in the field of numerical studies. Hence, it might not be very easy for us to encourage young students to get involved with time-consuming work on analytical perturbation theory, especially in the current "Publish or perish" atmosphere of the academic world. However, we must never let this field of research perish.

4.2. *Long-term dynamical stability of planetary motion*

Long-term dynamical stability of solar system planetary motion has been a central problem of celestial mechanics for centuries (cf. Tremaine, 1995). This problem has been debated over several hundred years since Newton's era, and has attracted many famous mathematicians over the years. This problem has also played a significant role in the development of nonlinear dynamics and chaos theory. However, we do not yet have a definite answer to the question of whether our solar system is stable or not. This is partly due to the fact that the definition of "stability" is vague when it is used in relation to the problem of planetary motion in the solar system. Nobili *et al.* (1989) fluently expressed this confusion in the introductory part of her paper:

> "When the stability of our solar system is discussed, two objections often arise. Firstly, this problem has been around for too long, never getting to the point of stating clearly whether the system is stable or not; the few definite results refer to mathematical abstractions such as N-body models and do not really apply to the real solar system. Secondly, the solar system is macroscopically stable—at least for a few 10^9 years—since it is still there, and there is not much point in giving a rigorous argument for such an intuitive property."

The traditional approach to dealing with this problem was to resort to analytical perturbation theories. But the recent progress of computer technology pushes us toward numerical studies. Numerical research on this problem started in the 1950s when electronic computing machines became available. Because of the very limited computational resources, the object of numerical integrations then was focused on the five outer planets [3] (Jupiter, Saturn, Uranus,

[3] As is already well known, Pluto has been "demoted" from the category of *planet* to *dwarf planet* by International Astronomical Union

Neptune, and Pluto) whose orbital timescales are much longer than those of the four inner planets; Mercury, Venus, the Earth, and Mars. The first famous numerical computation of planetary orbits was by Eckert *et al.* (1951) who did a 350-year simulation of the four outer planets (Jupiter, Saturn, Uranus, and Neptune) using a large mainframe computer. This was extended by Cohen and Hubbard (1965) to 120,000 years, and by Cohen *et al.* (1973) to one million years. These integrations agreed well with the perturbation calculations of Brouwer and van Woerkom (1950), showing quasiperiodic behavior for the four major outer planets. Later, Kinoshita and Nakai (1984) extended the length of the integration of the five outer planetary motion (Jupiter, Saturn, Uranus, Neptune, and Pluto) to five million years. One of the longest numerical integrations using conventional integration schemes on a general purpose computer was the LONGSTOP 1B integration done by Nobili *et al.* (1989) which ran for 100 Myr. Consult Roy *et al.* (1988) or Milani *et al.* (1987, 1989) for details of the LONGSTOP project. Their integrations followed the mutual interactions of the five outer planets, but also included the secular effect of the four inner planets and the effect of general relativity.

Integrations of the five outer planets for periods up to 845 Myr was done with a special-purpose computer, the Digital Orrery (Applegate *et al.*, 1986; Sussman and Wisdom, 1988). Sussman and Wisdom (1988) found in their long-term integration that the orbital motions of five outer planets including Pluto are chaotic. The Digital Orrery was a specialized but programmable high-performance computer designed for the efficient numerical integration of the equations of motion for systems with a small number of bodies that move in roughly circular orbits. This machine consisted of one CPU per planet, with all the CPUs arranged in a ring. Wisdom and Holman (1991) extended the length of the numerical integrations of five outer planetary motion to 1.1 Gyr using their original high-speed symplectic map. The integration period was further extended to ± 5.5 Gyr for the five outer planets by Kinoshita and Nakai (1995, 1996) using a PC with a dedicated numerical processor.

When we include four inner planets in the orbital integrations, the amount of computation is multiplied about 150 times: Stepsize of integration must be $P_{\rm Jupiter}/P_{\rm Mercury} \sim 50$ times smaller (where P is the orbital period of planets), and the number of force combinations becomes ${}_{9+1}\mathrm{C}_2/{}_{5+1}\mathrm{C}_2 = 3$ times larger. In addition, until recently the length of direct integrations was limited not by the CPU time, but by numerical errors. Milani and Nobili (1988) concluded that it was impossible to reliably integrate the orbits of the outer planets for a period of 10^9 years or more with the computer hardware and conventional algorithm current at that time. The main problem they found was the limited machine precision, and empirical evidence that the longitude error after n steps is proportional to n^2. Quinn and Tremaine (1990) proposed several corrections to conventional integration algorithms which have considerably reduced the roundoff error. Quinn *et al.* (1991) proposed a high-order multistep symmetric scheme. Quinn *et al.* (1991) utilized some of the numerical techniques that were state-of-the-art at the time to make an accurate integration of all nine planets and Earth's spin axis for 3.05 Myr into the past and future. Previously, Richardson and Walker (1989) performed 2 Myr integrations of all nine planets, but they neglected the effects of general relativity and the finite size of the Earth–Moon system.

After the publication of Sussman and Wisdom's (1992) work that confirmed that the Lyapunov time of the four inner planets is less than several million years, the boom in long-term numerical integration seemed to slow somewhat. But thanks to the recent availability of low-cost and high-performance PCs, much longer-term numerical integrations are now possible. Two of the longest numerical integrations currently published are Duncan and Lissauer (1998) and Ito and Tanikawa (2002). Duncan and Lissauer's (1998) main target was the effect of post-main-sequence solar mass loss on the stability of our planetary system. They performed many numerical integrations up to 10^{11} years of four jovian planets' orbital motion, and several 10^9 yr integrations of seven planets (from Venus to Neptune). The initial orbital elements of the planets are the same as those of our solar system planets, but the sun's mass decreases relative to those of the planets in their experiments. They found that orbit crossing times of planets, which can be a typical indicator of an instability timescale, are quite sensitive to the degree of the sun's mass decrease. The reduction of the sun's mass induces reduction of the instability, probably increasing the mutual Hill radii (also known as tidal radii; cf. Gladman (1993)) among planets. When the sun's mass is not so different from its present value, jovian planets remain stable over 10^{10} years, or perhaps more. Duncan and Lissauer (1998) also performed four similar experiments on the orbital motion of the seven planets from Venus to Neptune spanning about 10^9 years. It seems that the inner planets (Venus, the Earth, and Mars) also remain stable during the integration period, maintaining quasiperiodic oscillations similar to the present one.

as of August 24, 2006 (see http://www.iau.org/ for more detail). However in this manuscript, our treatment of Pluto goes along how original publications dealt with this body; Pluto has been treated as a planet in most previous literature. In addition, we are aware of the fact that the definition of "inner planets" or "outer planets" sometimes differs by who uses these terms. Generally in this section, "the five outer planets" denotes Jupiter, Saturn, Uranus, Neptune, and Pluto, "the four outer planets" or "the four jovian planets" denotes Jupiter, Saturn, Uranus, and Neptune, and "the four inner planets" means Mercury, Venus, the Earth, and Mars. We also try to list the planet names explicitly such as "the five outer planets (Jupiter, Saturn, Uranus, Neptune, and Pluto)" or "the four inner planets (Mercury, Venus, the Earth, and Mars)" whenever we think necessary.

Ito and Tanikawa (2002) presented the results of very long-term numerical integrations of planetary orbital motions over $\pm 5 \times 10^9$-yr timespan, encompassing all nine planets from Mercury to Pluto. Their numerical model includes only the classical Newtonian gravitation among planets and the Sun. Satellites, general relativity, solar mass loss, or other non-gravitational forces were neglected. Their numerical data shows that the planetary motion seems quite stable even for a 10^{10}-year timespan. A closer look at the lowest-frequency oscillations in their result indicates the potentially diffusive character of the four inner planetary motion, especially that of Mercury. The behavior of Mercury's orbital eccentricity in their integrations is qualitatively similar to that found in the results of Laskar's secular perturbation theory. However, there is no apparent secular increase in eccentricity or inclination in any planet's orbital elements. Ito and Tanikawa (2002) have also performed a couple of integrations including only the five outer planets for $\pm 5 \times 10^{10}$ years. They found that the three major resonances in the Neptune–Pluto system have been maintained over a 10^{11}-yr timespan, with rigorous stability in the orbital motion of all five outer planets.

The experience and knowledge gained through the long-term numerical study of our solar system's planets have been inherited by the new generation of researchers of extrasolar planetary systems. The overwhelming number of discoveries of extrasolar planets (more than 200 as of April 2007. See Butler *et al.* (2006) for more detail) suggests that one of our future tasks is to count and categorize the mutual interactions of planets in our solar system as well as in other planetary systems, for as many timescales as possible: a taxonomy of planetary stability and instability. This will be an important theme of solar system dynamics in the near future, and will form a basis for the dynamical study of extrasolar planetary systems. In this sense, a set of numerical experiments by Lissauer *et al.* (2001) is quite interesting and intriguing. If a planetary mass body were present in the asteroid belt, the orbits of the four inner planets and those of the outer giant planets would be more closely coupled. A greater exchange of angular momentum could affect the stability of the four inner planets. More extensive and more comprehensive studies in this line should be done for a larger range of parameters of various planetary systems.

4.3. *Dynamics in planet formation study*

Equipped with the evidence that the current solar system planetary motion is stable for a very long time, research into the origin and evolution of planetary systems has progressed remarkably with the help of celestial mechanics. This field involves celestial mechanics mainly in terms of the dynamical motion of planetesimals (or protoplanets), created by the condensation of dusts in protoplanetary nebula (e.g. Hayashi *et al.*, 1985). Celestial mechanics becomes more important in the latter stages of planet formation when we can approximate the dynamics of particles (planetesimals or protoplanets) as that of point masses. One of the biggest problems of the so-called "standard model" of planet formation was that the accretion timescales of jovian planets, especially those of icy Uranus and Neptune could be very long, even longer than the age of the solar system itself. In the 1980s, efforts to solve this problem have led to investigation of the runaway growth of planetesimals (e.g. Wetherill and Stewart, 1989).

As for the dynamical research in this field, we should mention the enormous success of the special-purpose supercomputer for the gravitational N-body problem, GRAPE (see Section 5.4 for detail). Fully exploiting the ultra-high computational speed of the GRAPE system, S. Ida and E. Kokubo succeeded in performing a direct calculation of planetary accretion from planetesimals to protoplanets. According to their series of papers (Kokubo and Ida, 1995, 1996, 1998, 2000, 2002), the mode of planetesimal accretion is inevitably "runaway" under the physical and dynamical conditions of the protosolar nebula. Swarms of planetesimals are accumulated and grow oligarchically into much fewer number of protoplanets (or planetary embryos). Now the concept of oligarchic growth is also common to the formation of giant planets (e.g. Thommes *et al.*, 2003).

There is an interesting dynamical feature of the final stage of planet formation. When we look at the relationship between the separation of protoplanets and their stability, we can see a kind of an experimental scaling law. It is a relationship between the normalized distance between protoplanets and time until initial instability of the system. This relationship was first confirmed by Chambers *et al.* (1996) based on a previous research by Gladman (1993). Chambers *et al.* (1996) found that the time T_E, until the first close encounter between two protoplanets, is exponentially proportional to the protoplanetary distance Δ normalized by the mutual Hill radii; $T_E \propto e^{\Delta}$. They integrated systems of several protoplanets with equal-mass, initially on equally spaced coplanar and circular orbits. This relationship was later confirmed also by Yoshinaga *et al.* (1999) when the orbits of the protoplanets are elliptic and inclined, and by Ito and Tanikawa (1999, 2001) when Jupiter is outside the protoplanetary system as a perturber. In Ito and Tanikawa's (1999) numerical experiments, this exponential law stands over five orders of magnitude in T_E. However, the dynamical fundamentals of this relationship are still unknown, though it is expected that they are based on a certain chaotic diffusion in phase space.

A similar but different scaling relationship is known to exist between the mass and the instability timescale of planet-like bodies. Duncan and Lissauer (1997) integrated systems with initial orbits that are identical to those of the satellites of Uranus. In their experiments, the masses of the satellites are all increased from actual values by the same factor $m_{\rm f}$. They found that the orbital crossing time T_C obeyed a relationship as $T_C \propto m_{\rm f}^{\alpha}$ where α is a negative constant. These power law fits even extend over seven orders of magnitude in the crossing time T_C. The theoretical fundamental of the relationship $T_C \propto m_{\mathrm{f}}^{\alpha}$ is not known either. The two scaling relationships, $T_E \propto e^{\Delta}$ and $T_C \propto m_{\mathrm{f}}^{\alpha}$, may be two special cases of a more general relationship, and may be involved with the relationship between Lyapunov time T_L and real instability time T_I that we mentioned in Section 2.2: $T_I \propto T_L^{\gamma}$ in Eq. (13).

Let us observe the current solar system planets from the standpoint of normalized planetary separation, Δ. Δ between terrestrial planets is very large, such as $\Delta > 26R_H$ where R_H is the mutual Hill radius of planets. On the other hand, those separations among jovian planets are much smaller, such as $\Delta < 14R_H$. Although this difference could be one of the essential factors in maintenance of the solar system stable planetary motion over billions of years (cf. Ito and Tanikawa, 1999, 2002), the reason of this difference is yet to be precisely known. There have been some efforts to explain the origin and difference of the planetary separations as being caused by dynamical evolution after the formation of the planetary system (e.g. Laskar, 1997), but it would be more natural to ascribe these differences to the formation process of the planets. The results of N-body numerical simulations of the runaway growth of planetesimals tell us that the initial value of the normalized separation of protoplanetary systems could be $\Delta = 5$–$10R_H$ (Kokubo and Ida, 1998). So the current separations between the jovian planets are consistent with what N-body integration predicts. The problem left is that of when and how the normalized separation between terrestrial planets became so large over the course of their growth from protoplanets. This question naturally leads us to the study of the late stage of terrestrial planet formation. A large number of papers have been, and will be, published along this line of research, employing accurate and long-term N-body integrations such as Chambers and Wetherill (1998), Levison *et al.* (1998), Agnor *et al.* (1999) Chambers (2001), Kominami and Ida (2002, 2004), Kominami *et al.* (2005), and Daisaka *et al.* (2006). Together with dynamical studies of the diversity of extrasolar planetary systems, this field will remain one of the major research areas of planetary dynamics in the next decade.

As well as the late stage of planet formation where planetesimals and protoplanets play a central role, the very beginning stage of planet formation is also dominated by a certain kind of dynamics: the dynamics of dust. Dust is one of the key ingredients of the solar system. As well as gas, the dynamics of dust controls the early stage of planetary formation toward the accretion of planetesimals. However, the dynamics of dust is somewhat different from that of larger bodies, particularly in connection to the solar radiation. Dust particles are so small that they experience solar radiation and stellar wind forces. These pressures change the orbital elements of dust particles a great deal. Dust particles even escape from the system on hyperbolic orbits because of the additional angular momentum and orbital energy from the radiation pressure. The Poynting–Robertson effect and solar wind drag tend to circularize their orbits, forcing the dust particles to slowly drift toward the central star (Burns *et al.*, 1979). As for a more detail of the dynamics of dust and its significance in the planet formation, consult publications such as Backman and Paresce (1993), Liou *et al.* (1995), Stern (1996), Yamamoto and Mukai (1998), and Moro-Martín and Malhotra (2002, 2003).

4.4. *Dynamics of small bodies*

Dynamics of minor bodies in our solar system always provides numerous interesting exercises for celestial mechanics. Now that we know of a huge number of small bodies in our solar system, the number of dynamical phenomena that requires celestial mechanics for explanation is soaring. Recent reviews such as Lazzaro *et al.* (2006) will be a good guide for readers to know how extensively and fast the research of this field goes on. Among many kinds of minor body population research, particularly we focus on the following topics in this subsection: asteroids and resonant dynamics, the Yarkovsky effect and asteroid family, and trans-Neptunian objects including the Kuiper Belt and the Oort Cloud objects.

4.4.1. *Asteroids and resonant dynamics*

More than two hundred years have passed since the first discovery of an asteroid (Ceres: discovered in 1801), and almost ninety years have passed since the discovery of the Hirayama asteroid families (Hirayama, 1918). More than 360,000 asteroids have been discovered as of April 2007, and a vast amount of knowledge has been accumulated about asteroids; knowledge of spatial distribution, spectroscopic information, taxonomy, shape, rotational motion, and orbital dynamics. In terms of long-term orbital dynamics, asteroids are particularly interesting from the standpoint of their complicated resonant behavior in relation to the existence of numerous Kirkwood gaps in the main asteroid belt (e.g. Murray and Dermott, 1999).

Mean motion and secular resonances

Two sorts of resonances that prevail in asteroid dynamics are mean motion resonances and secular resonances. A mean motion resonance occurs when two bodies have orbital periods in a commensurability (a simple integer ratio, such as 1:2). In the main asteroid belt within 3.5 AU from the sun, the major mean motion resonances with Jupiter form gaps in the asteroid distribution such as at 3:1, 5:2, 7:3, or 2:1 resonances. They are the Kirkwood gaps (Kirkwood, 1867). Most asteroids have been ejected from these zones by repeated encounters with Jupiter. On the other hand in the outer main asteroid belt, 3:2, 4:3 and 1:1 resonances with Jupiter are populated by groups of asteroids: the Hilda group (3:2), the Thule group (4:3), and the Trojans (1:1). Dynamics of mean motion resonances is well illustrated and explained in Murray and Dermott (1999).

A secular resonance arises when the rate of change of the proper longitude of pericenter or of proper longitude of ascending node of a test body and one of the eigenfrequencies of the system of perturbing bodies are in a commensurability. The location of secular resonances in the solar system (known under the names such as ν_5, ν_6, or ν_{16}) is rather complicated because of the coupling of terms related to eccentricities and those related to inclinations in the disturbing function. The location corresponds to surfaces in three-dimensional (a, e, I) space where a is the semimajor axis, e is the eccentricity, and I is the inclination. Hence secular resonances form the boundaries of the main asteroid belt in the (a, e, I) space. For more detail about the theories of secular resonances, consult publications such as Williams (1969, 1979), Williams and Faulkner (1981), Scholl *et al.* (1989), Morbidelli and Henrard (1991a,b), Milani and Knežević (1990), Froeschlé and Morbidelli (1994), Knežević and Milani (1994), Knežević *et al.* (1995), and Murray and Dermott (1999).

In the early 1980s, J. Wisdom gave asteroid dynamists a huge shock. He introduced a mapping method to simulate asteroid dynamics, exploiting Dirac's δ-function (Wisdom, 1982, 1983). In his map, the continuous action of force is replaced by impulses, resulting in a drastic shortening of the computation time. Wisdom observed that asteroid orbits in the 3:1 mean motion resonance attain a high eccentricity and may collide with Mars or the Earth. This method and result were initially regarded with some doubt, but soon confirmed and recognized as correct. The speed of his mapping method revolutionized the study of asteroid dynamics. Before Wisdom, a one-million-year integration of an asteroid orbit was a tremendous task. With the map derived from the planar elliptic restricted three-body problem, Wisdom found that the orbital eccentricity of the small body can suddenly increase after a long time of quiescence. In addition, it turned out that the increase and decrease in eccentricity is repeated chaotically. This was the first work to explicitly deal with solar system chaos. Wisdom's result was later confirmed by himself through direct integrations of the equations of motion (Wisdom, 1987b).

After Wisdom, a large number of celestial mechanists have struggled through an enormous number of publications to explain the origin of the Kirkwood gaps at various resonances. For example, Yoshikawa (1987, 1989), within the framework of the planar (and partially three-dimensional) elliptic restricted three-body problem, semi-analytically considered the motion of asteroids in 3:1, 5:2, 7:3, and 2:1 resonances. He also carried out numerical integrations and illustrated the orbital changes in mean motion resonances. Morbidelli and Moons (1993) and Moons and Morbidelli (1995), in the restricted four-body problem of the Sun, Jupiter, Saturn, and an asteroid, studied secular resonances in 3:2 and 2:1 mean motion resonances three-dimensionally and 4:1, 3:1, 5:2, and 7:3 mean motion resonances. They mapped effective places of secular resonances in the region of the mean motion resonance. As an example, in the 3:2 resonance, ν_5 and ν_6 occupy large chaotic areas, whereas ν_{16} resonance works in the 2:1 resonance.

The presence of Saturn introduces a number of additional frequencies into Jupiter's orbit, which can significantly influence the orbits of asteroids. The resonant arguments of these three-body resonances contain the longitudes of Jupiter and asteroid together with either the secular frequency g_6, or the longitude of Saturn. Resonances involving the longitude of Saturn are analogs of the Laplace resonance in the Jovian satellite system (e.g. Greenberg, 1977). Murray *et al.* (1998) considered the effect of gravitational perturbations from Jupiter on the dynamics of asteroids, when Jupiter is itself perturbed by Saturn. Murray *et al.* (1998) showed that many three-body resonances involving the longitude of Saturn are chaotic, and they gave simple expressions for the width of the chaotic region and the associated Lyapunov time.

Kozai mechanism

In addition to ordinary secular and mean motion resonances, Kozai (1962b) investigated the long-term motion of asteroids with high inclination and eccentricity. As we mentioned in Section 3.5, Kozai found what we now call the Kozai mechanism (or the Kozai behavior) that drives the eccentricity and the inclination of asteroids very high, such as $e \sim 1$ or $I \sim 90°$ under certain conditions. Kozai (1962b) is one of the pioneers in exploiting equi-energy curves (equi-Hamiltonian maps) with one degree of freedom for the purpose of investigating the global dynamical behavior of test particles undergoing the perturbation of massive bodies. The concept of the Kozai mechanism was extended by Michel and Thomas (1996) to include the four jovian planets. Michel and Thomas (1996) showed that the Kozai mechanism provides a temporary protection mechanism for near-Earth asteroids from close planetary encounters. A near-Earth asteroid can be locked in the Kozai mechanism in a stable state, where it can stay for a comparatively long time before a close approach with a planet drastically changes its orbit. Now the concept of the Kozai mechanism is ubiquitous throughout celestial mechanics, not only in the asteroidal motion field, but in the study of the motion of Kuiper Belt objects (e.g. Duncan *et al.*, 1995; Kuchner *et al.*, 2002), comets (e.g. Bailey *et al.*, 1992; Bailey and Emelýanenko, 1996), Pluto (e.g. Williams and Benson, 1971; Wan *et al.*, 2001), natural and artificial satellites (e.g. Giacaglia *et al.*, 1970; Mignard, 1975; Nesvorný *et al.*, 2003), meteoroids (e.g. Wetherill, 1974; Valsecchi *et al.*, 1999), binaries and triple stars (e.g. Innanen *et al.*, 1997; Mikkola and Tanikawa, 1998a; Mardling and Aarseth, 2001), forming planets (e.g. Chambers and Wetherill, 1998; Levison *et al.*, 1998; Levison and Agnor, 2003), extrasolar planets (e.g. Laughlin and Adams, 1999; Ford *et al.*, 2000;

Drovak *et al.*, 2003), and even supermassive binary black holes (e.g. Blaes *et al.*, 2002). Gronchi and Milani (1999b) devised a semi-analytical method to extend the application of the Kozai mechanism to planet-crossing orbits.

Secondary resonances

The dynamics of asteroids is characterized not only by mean motion resonances with Jupiter or simple secular resonances, but by complicated secondary resonances and resonance overlaps. For example, when we consider the planar circular restricted three-body problem, the resonance can be classified into primary and secondary resonances. Primary resonances are due to commensurabilities between the mean motions of the perturbing body and the test particle. Secondary resonances occur when the libration frequency of a primary resonance is commensurate with one of the higher-order primary resonances (Murray and Dermott, 1999). Secondary resonances are not just a product of theory, but an actual phenomenon that can have a significant effect on the asteroid and satellite dynamics. We can see a typical example of secondary resonances in the orbit of a Uranian satellite, Miranda. The series of works by R. Malhotra, S. F. Dermott, and C. D. Murray (e.g. Dermott *et al.*, 1988; Malhotra and Dermott, 1990; Malhotra, 1990) on the role of secondary resonances in the orbital history of Miranda established an important basis for the dynamical treatment of secondary resonances in solar system dynamics, and is concisely summarized in Kortenkamp *et al.* (2004). Malhotra and her colleagues showed that the anomalously large orbital inclination of Miranda is naturally explained as a consequence of the secondary-resonance sweeping due to the tidal evolution within a primary 3:1 inclination-type mean motion resonance with Umbriel, another satellite of Uranus. The primary resonance increased Miranda's inclination from an initially small value, but the resonance was temporary. As Miranda's inclination approached its current value, the satellites were captured in a secondary resonance which amplified their primary-resonance libration amplitude. This eventually caused the satellites to escape from the 3:1 mean-motion resonance, leaving Miranda with an inclination that is preserved to the present. The secondary resonance implicated in this case was due to a 1:3 commensurability between the libration frequency of the primary mean motion resonance angle and the secular frequency of precession of the relative lines of nodes of the two satellites.

Another famous example of secondary resonances in the solar system is seen in the motion of Pluto, particularly in its resonant relation with Neptune. Pluto's orbital motion is characterized by three major resonances with Neptune (Kinoshita and Nakai, 1995, 1996) which ensure that Pluto and Neptune do not encounter each other: (i) Pluto and Neptune are in a 2:3 mean motion resonance with the critical argument $\theta_1 = 3\lambda_P - 2\lambda_N - \varpi_P$. (ii) Pluto's argument of perihelion $\omega_P = \theta_2 = \varpi_P - \Omega_P$ librates around $90°$ with a period of about 3.8×10^6 years. This is a typical example of the Kozai mechanism. (iii) The longitude of Pluto's node referred to the longitude of Neptune's node ($\theta_3 = \Omega_P - \Omega_N$) circulates, and the period of θ_3 circulation is equal to the period of θ_2 libration. The third resonance, sometimes called the "superresonance" (Milani *et al.*, 1989; Wan *et al.*, 2001), is a 1:1 secondary resonance between θ_2 and θ_3.

Let us give some examples of the study of secondary resonances in the asteroidal motion. Moons and Morbidelli (1993) studied the motion of asteroids in the mean motion commensurabilities through the planar restricted three-body problem. They provided global pictures of the dynamics in the region of secondary resonances. Ferraz-Mello (1994) studied the structure of the phase space of the 2:1 resonance through the planar averaged asteroidal three-body problem. The only chaotic regions he found were those associated with secondary resonances, confined to low eccentricities, and those associated with large libration amplitudes. Nesvorný and Ferraz-Mello (1997) applied Laskar's (1993) frequency map analysis to the dynamical models of the 2:1 asteroidal mean motion resonance; the planar restricted three-body model and the planar restricted four-body model with Saturn as the third primary. Their result reproduced the chaotic region formed by the overlap of secondary resonances in low eccentricities. Nesvorný and Ferraz-Mello (1997) examined the chaos generated by high-order secondary resonances in moderate eccentricities.

Secondary resonances might have worked as a clearing mechanism of trojan-type companions of Neptune during primordial migration of four giant planets (Kortenkamp *et al.*, 2004). The loss of Neptune trojans (which we know very few of now) can occur when trojan particles are swept by secondary resonances associated with mean motion commensurabilities of Uranus with Neptune (2:1). These secondary resonances arise when the circulation frequencies of critical arguments for Uranus–Neptune mean motion near-resonances are commensurate with harmonics of the libration frequency of the critical argument for the Neptune–trojan 1:1 mean motion resonance. Trojans that are trapped in the secondary resonances typically have their libration amplitudes amplified until they escape the 1:1 resonance with Neptune. Trojans with large libration amplitudes are susceptible to loss during sweeping by numerous high-order secondary resonances.

Resonance overlap

In general, a resonance can have its own finite width. This finite width can cause an overlap of nearby resonances, which frequently results in the onset of chaos. As B. V. Chirikov predicted nearly 30 years ago (Chirikov, 1979), this resonance overlap plays a significant role in the dynamical behavior of asteroids. Right after Chirikov (1979), Wisdom (1980) featured the resonance overlap in relation to the onset of stochastic behavior of the planar circular restricted three-body problem. After that, the concept of resonance overlap was gradually recognized and accepted by the society of solar system dynamics. For example, Froeschlé and Scholl (1989) performed numerical experiments that indicated possible chaotic motion due to overlapping resonances in the asteroid belt. A secular resonance may overlap with another secular resonance or with a mean motion resonance. Morbidelli and Moons (1995) carried out numerical integrations of objects that are in the 3:1 mean motion resonance. Their result indicates that the dynamics in the 3:1 resonance zone is strongly chaotic, due to the overlapping of secular resonances. As a consequence, the eccentricity of bodies in this zone can increase approximately to unity on a time scale of one million years. Holman and Murray (1996) investigated the origin of chaos near high-order mean motion resonances in the outer asteroid belt. They surveyed the variation of the Lyapunov time with semimajor axis throughout the outer belt region with the elliptic restricted three-body problem. They also developed an analytic theory based on resonance overlap that predicts the widths in semimajor axes as well as typical Lyapunov times of the chaotic zones associated with these resonances. Michel and Froeschlé (1997) presented the location of linear secular resonances in the region of semimajor axes a less than 2 AU where many near-Earth asteroids are found. The positions of the secular resonances in the plane (a, I) show that the inner solar system is dynamically very complex: a lot of resonances are present, and some of them overlap. An extensive review has been done by Lecar *et al.* (2001) for the relationship between asteroidal resonances and chaos, including resonance overlaps.

Trojans and migration of resonances

Talking about resonances, minor bodies around the 1:1 mean motion resonance with major planets, Trojan-type companions or co-orbital objects, still provide us with a big challenge in terms of celestial mechanics, especially as a realistic application of the restricted three-body problem. As of April 2007, more than 2000 asteroids are known to share Jupiter's orbit, five others are known as Neptune trojans, but none is known around Saturn and Uranus. The origin of this difference is not yet fully understood, though several hypotheses have been proposed. Consult the introduction of Kortenkamp *et al.* (2004) and the references therein. Also, recent discoveries of co-orbital objects around the Earth (Wiegert *et al.*, 1997, 1998, 2000; Connors *et al.*, 2002) and Venus (Mikkola *et al.*, 2004; Brasser *et al.*, 2004; Morais and Morbidelli, 2006) have prompted a number of questions about their origin and stability.

Among many theories and hypotheses as to the origin of Jupiter's trojans, a series of papers by A. Morbidelli, R. Gomes, K. Tsiganis, and H. F. Levison are of particular interest. In previous literature on asteroid dynamics, Jupiter's trojans have most frequently been considered to be planetesimals that formed near Jupiter and were captured onto their current orbits while Jupiter was growing (e.g. Marzari and Scholl, 1998; Fleming and Hamilton, 2000), possibly with the help of gas drag and/or collisions (e.g. Shoemaker *et al.*, 1989). However, this hypothesis cannot explain some basic properties of the trojan population, in particular its broad orbital inclination distribution, which ranges up to $\sim 40°$ (Marzari *et al.*, 2002). Morbidelli *et al.* (2005) showed that Jupiter's trojans could have formed in more distant regions and been subsequently captured into co-orbital motion with Jupiter when the giant planets migrated by removing neighboring planetesimals. The capture was possible during a short period of time, just after Jupiter and Saturn crossed their mutual mean motion 2:1 resonance, when the dynamics of the trojan region were completely chaotic.

Morbidelli *et al.*'s (2005) simulations of the process involving the resonance passing of Jupiter and Saturn do not only satisfactorily reproduce the orbital distribution of the trojans and their total mass, but it can reasonably account for an intense period of planetesimal bombardment in the inner solar system, collectively called the "late heavy bombardment" or the LHB. The petrology record on the Moon suggests that a cataclysmic spike in the cratering rate occurred about 700 million years after the planets formed (Tera *et al.*, 1974; Ryder, 1990; Hartmann *et al.*, 2000). Planetary formation theories cannot naturally account for a bombardment period so late in the solar system history (Morbidelli *et al.*, 2001; Levison *et al.*, 2001a). Using the same dynamical model as Morbidelli *et al.* (2005), Gomes *et al.* (2005) proposed that the LHB was triggered by the rapid migration of the giant planets, which occurred after a long quiescent period. During this burst of migration, the planetesimal disk outside the orbits of the planets was destabilized, causing a sudden massive delivery of planetesimals to the inner solar system. The asteroid belt was also strongly perturbed, with these objects supplying a significant fraction of the LHB impactors in accordance with recent observational evidence (e.g. Bogard, 1995; Cohen *et al.*, 2000; Kring and Cohen, 2002; Strom *et al.*, 2005; Ito and Malhotra, 2006).

Incidentally, the dynamical model used by Morbidelli *et al.* (2005) and Gomes *et al.* (2005) also explains well the current values of eccentricities of major giant planets; Jupiter, Saturn, and Uranus. Again using the same dynamical model, Tsiganis *et al.* (2005) showed that a planetary system with initial quasi-circular, coplanar orbits would have evolved to the current solar system planetary orbital configuration, provided that Jupiter and Saturn crossed their 2:1 mean motion resonance. Tsiganis *et al.* (2005) showed that this resonance crossing could have occurred as the giant planets migrated owing to their interaction with a disk of planetesimals (Fernández and Ip, 1984; Malhotra, 1995). Tsiganis *et al.*'s dynamical model reproduces all the important characteristics of the giant planets' orbits; their final semimajor axes, eccentricities and mutual inclinations. In addition, it turned out that this migration scenario suggests that the present obliquities of the giant planets were presumably achieved when Jupiter and Saturn crossed the 2:1 mean motion resonance (Brunini, 2006). The existence of the regular satellites of the giant planets does not represent a problem in this scenario because, although they formed soon after the planet formation, they can follow the slow evolution of the equatorial plane of planets during the migration stage.

4.4.2. *The Yarkovsky effect and asteroid families*

A group of asteroids that have similar proper orbital elements are called an asteroid family. The concept of the asteroid families, i.e. *the Hirayama asteroid families,* was first advocated by K. Hirayama (1918), followed by numerous research efforts. Hirayama found three families in his celebrated 1918 paper: Koronis, Eos, and Themis families. The Hirayama asteroid families are considered remnants of collisional disruption of Myr to a few Gyr ago (Zappalà *et al.*, 1994). The proper elements are analytically defined as constants of motion of a suitably simplified dynamical system, obtained by classical transformation theory from a real dynamical system (Milani and Knežević, 1990; Morbidelli, 2002; Knežević *et al.*, 2002). If the real dynamics is regular, proper elements show only very small oscillations around a mean value that stays constant with time (Knežević *et al.*, 1995). However, if the motion is chaotic, the mean value of the proper elements can drift in time at a rate that depends on the properties of the chaotic motion (Milani and Farinella, 1994). Now it has turned out that the proper elements could be altered, and asteroid families are diffused a great deal over the long timescale of 100 Myr to Gyr, owing to numerous weak resonances (Morbidelli and Nesvorný, 1999), to mutual close encounters with massive asteroids (Nesvorný *et al.*, 2002), as well as to the Yarkovsky thermal effect (Bottke *et al.*, 2001; Nesvorný *et al.*, 2002).

The Yarkovsky effect is caused by a thermal radiation force that makes objects undergo slow but steady semimajor axis drift and spin-up/down as functions of their rotation, orbit, and material properties (Rubincam, 1995, 1998). Numerical results suggest that this mechanism can be used to: (i) deliver asteroids with diameter $D \lesssim 20$ km from their original locations in the main belt to resonance regions that are capable of transporting them to Earth-crossing orbits, (ii) disperse asteroid families, with drifting bodies jumping or becoming trapped in mean-motion and secular resonances within the main belt, and (iii) modify the rotation rates of asteroids a few km in diameter or smaller.

The Yarkovsky effect solves several definite inconsistencies that gravitational forces cannot explain in terms of asteroid dynamics. For example, the meteorite cosmic-ray exposure age of asteroids is an order of magnitude longer than conventional model predictions. The mechanism for continuously supplying near-Earth asteroids to the mean motion resonance zones in the main belt is not well explained, either (Bottke *et al.*, 2002). Since the phenomena in the solar system are not always "clean" (i.e. expressed only by the law of gravity), sometimes we have to extend the applicable area of celestial mechanics from what ordinary gravity law governs to something that simple gravity law cannot cope with, especially in asteroid dynamics. The Yarkovsky effect is a typical and very important one of these "dirty" forces.

The Yarkovsky effect should now be considered as being as important as collisions and gravitational perturbations are to our overall understanding of asteroid dynamics, especially that of the Hirayama asteroid families. The orbital distributions of prominent families are thought to be by-products of catastrophic disruption events. But ejection velocities derived from the orbital elements of families are much higher compared with the results of impact experiments and simulations. One way to resolve this contradiction is to assume that $D \lesssim 20$ km family members have undergone semimajor axis drift due to the Yarkovsky effect. W. Bottke demonstrated how the Yarkovsky effect could influence creation of some of the existing Hirayama asteroid families (Bottke *et al.*, 2001). Their result explains why families are sharply bounded by nearby Kirkwood gaps, why some families have asymmetric shapes, and the curious presence of family members on short-lived orbits.

The Yarkovsky effect also influences the rotational motion of asteroids. The Yarkovsky–O'Keefe–Radzievskii–Paddack (YORP) effect, a version of the Yarkovsky effect, can spin up or down asteroids with 10 km diameter on a 10^8-year timescale (Rubincam, 2000). Smaller asteroids spin up or down even faster due to the radius-squared dependence of the YORP timescale. The YORP effect may explain the rapid rotation, slow tumbling, and spin-orbit coupling appearing in some asteroid rotational motions.

Recently, the Yarkovsky and the YORP effects have even been directly detected by accurate observations. Vokrouhlický *et al.* (2000) argued that a precise radar refinement of the orbits of near-Earth asteroids offers the possibility of detecting the Yarkovsky effect during the next few decades. Chesley *et al.* (2003) reported the prospects for the detection of the Yarkovsky effect from fine-precision Doppler radar observations of the half-kilometer asteroid, (6489) Golevka. Lowry *et al.* (2007) presented precise optical photometric observations of a small near-Earth asteroid, (54509) 2000 PH5, acquired over 4 years. They found that the asteroid has been continuously increasing its rotation rate ω over this period by $d\omega/dt = 2.0(\pm 0.2) \times 10^{-4}$ degrees per day squared. They simulated the asteroid's close Earth approaches from 2001 to 2005, showing that gravitational torques cannot explain the observed spin rate increase. Dynamical simulations suggest that 2000 PH5 may reach a rotation period of ~ 20 seconds toward the end of its expected lifetime. Also, Taylor *et al.* (2007) showed that radar and optical observations revealed that the continuous increase in the spin rate of this asteroid can be attributed to the YORP effect. The change in spin rate is in reasonable agreement with theoretical predictions for the YORP acceleration of a body with the radar-determined size, shape, and spin state of 2000 PH5. The detection of asteroid spin-up supports the YORP effect as an explanation for the anomalous distribution of spin rates for asteroids under 10 kilometers in diameter and as a binary formation mechanism. More accurate observation results will reveal the true nature of the Yarkovsky effect, which also gives us significant information about the dynamical history of asteroids as well as of other small bodies in the solar system.

4.4.3. *Kuiper Belt objects*

Owing to the recent development of high-accuracy observations, physical and dynamical research into the outermost area of our solar system seems to be approaching its zenith. Small bodies in the outermost area of the solar system that orbit the sun at distances farther than Neptune are collectively called trans-Neptunian objects (TNOs). The Kuiper Belt, the scattered disk (which is occasionally included in the Kuiper Belt), and the Oort Cloud are the names for three divisions of this volume of space.

Among the trans-Neptunian orbital zones, the Kuiper Belt is the innermost region: an area extending from the orbit of Neptune to $r \sim 50$ AU (r denotes heliocentric distance). Although the existence of the objects in this region had been predicted since the 1940s to the 1950s (Edgeworth, 1949; Kuiper, 1951), it took more than 40 years for the first object to be discovered (Jewitt and Luu, 1993). Since then, more than 1000 Kuiper Belt objects (KBOs) have been found, as of April 2007. The recent status of the KBO research is summarized in Weissman (1995), Jewitt (1999), Luu and Jewitt (2002), and Gladman (2005).

KBOs have been dynamically categorized into three types (Trujillo, 2003). About half of the KBOs have nearly circular orbits ($e < 0.2$) and form a kind of ring structure at around 42–48 AU. These are called "classical" objects. About 10% of the KBOs are in 2:3 mean motion resonance with Neptune, as Pluto is. These are called "plutinos". The rest of the KBOs, about 40% of them, have highly eccentric orbits, and are therefore called "scattered" objects. The orbits of the scattered KBOs are distributed very widely from the inner part of the classical object zone to the furthest area, at around 900 AU. Since the orbital characteristics of the scattered objects are so different from that of classical objects and plutinos, the space volume that scattered objects occupy is recently dubbed "the scattered disk", and is regarded as a separate category among the TNOs.

In spite of the numerous research efforts devoted to KBOs, detailed dynamical structure and origin of the KBOs are not well known yet. First of all, there seems to be a relatively sharp outer boundary for the KBOs, especially the classical Kuiper Belt (cf. Allen *et al.*, 2002). Beyond $a = 48$AU, no KBO has a circular orbit. A few explanations have been proposed to account for the formation of this edge, but none of them is widely accepted yet. One theory claims that the outermost particles were pretty much dynamically excited and stripped away by a close encounter with a passing star in the early stages of solar system history (e.g. Ida *et al.*, 2000; Kobayashi and Ida, 2001). Another says that an Earth-sized planet(s) may have passed through the primordial Kuiper Belt and pumped up the eccentricities and inclinations of the particles there (e.g. Petit *et al.*, 1999). Yet another possibility is that the primordial solar nebula and its dispersal caused the migration of some strong secular resonances outside of Neptune, which enhanced the random velocity of a certain part of the KBOs and created the edge (e.g. Nagasawa and Ida, 2000).

In addition to the existence of the outmost edge of the classical KBOs, recent observations (e.g. Burns, 2002; Veillet *et al.*, 2002; Noll *et al.*, 2002; Stephens and Noll, 2006) have shown that the fraction of binaries among the KBOs is unexpectedly high, much higher than the fraction of asteroid binaries in the asteroid belt. Also, the orbital separation of KBO binaries is usually larger than that of asteroid binaries, and their mass ratio is closer to unity (Veillet *et al.*, 2002). To theoretically understand these characteristics of KBO binaries, Funato *et al.* (2004) showed that the fraction of the KBO binary population was initially as small as that of the current asteroid binaries, but was gradually increased through three-body dynamical interactions. This was possible because the tidal force from the Sun is weaker in the Kuiper Belt area than in the asteroid belt. Different explanations have been attempted by Weidenschilling (2002) and Goldreich *et al.* (2002).

Some of the important dynamical characters of the KBOs probably have their origins in their formation stage. One of the most intriguing ideas that are derived from the dynamical characteristics of the KBOs is that the current orbital distribution of the KBOs was produced by the so-called resonance capture mechanism, triggered by the orbital migration of Neptune (and probably other three giant planets). This idea was first explored extensively by Malhotra (1993) to explain the origin of Pluto's orbit. Pluto's orbit is quite unusual compared with other planetary orbits, being the most eccentric and the most inclined. The orbits of Pluto and Neptune overlap. But close approaches of these two planets are prevented by the existence of a resonance condition such that Pluto's orbital period is exactly 3:2 that of Neptune. This ensures that the conjunctions always occur near Pluto's aphelion. Malhotra showed that Pluto could have acquired its current orbit during the late stages of planetary accretion and the formation of the Oort Cloud comets, when the jovian planets underwent significant orbital migration as a result of encounters with residual planetesimals. As Neptune moved outwards, a small body like Pluto in an initially circular orbit could have been captured into the 3:2 mean motion resonance. After this event, its orbital eccentricity would rise rapidly to its current Neptune-crossing value. If this mechanism occurred, the entire region from the orbit of Neptune to $r \sim 50$ AU would have been swept by strong mean motion resonances. This means that the resonance capture could have occurred not only for Pluto but for other KBOs. Malhotra's (1995) classic paper dealing with this theory explains some of the major consequences in the Kuiper Belt, such as the fact that most objects in the region beyond Neptune and up to $r \sim 50$ AU are confined in the narrow resonance zone, especially in 3:2 and 2:1 mean motion resonances.

A vast amount of literature has followed Malhotra's pioneering work of planetary orbital migration and resonant capture, not only in the area of KBOs formation research, but covering almost all stages of planetary formation. One of the recent reports has come from Levison and Morbidelli (2003). They have shown that the objects currently observed in the dynamically cold Kuiper Belt were most probably formed within ~ 35 AU, and were subsequently pushed outward by Neptune's 2:1 mean motion resonance during its final phase of migration. They have concluded that the entire Kuiper Belt formed closer to the Sun and was transported outward during the final stages of planet formation.

All these questions, hypotheses, and enigmas on the dynamical characteristics of the KBOs as well as of the scattered objects should be, and will be confirmed or eliminated by more accurate observations of distant regions such as $r > 50$ AU (Brown *et al.*, 2004). Accordingly, discoveries of very large KBOs or scattered objects such as (90377) Sedna or (136199) Eris will expand the outmost region the outer solar system to a great deal (e.g. Brown *et al.*, 2005). Actually, the discovery of Eris led astronomers and planetary scientists to question the definition of the term *planet,* as Eris turned out be larger than Pluto. This epochmaking finding eventually ended up with the demotion of Pluto from a planet to an object among the new category of small body population in the solar system: *dwarf planet.* The dynamical characteristics and origin of the outermost dwarf planets, such as Eris or Sedna, are now the subjects of intense research, and will be one of the central topics of the solar system dynamics in the new century.

Speaking of KBOs, though it seems digressive to talk about astronomical engineering in this manuscript, the idea of gradually expanding the Earth's orbit artificially using a KBO over a very long timespan (Korycansky *et al.*, 2001) is more than just an SF story, but it is an interesting thought experiment from the standpoint of the KBO and planetary dynamics.

4.4.4. *The Oort Cloud*

The existence of a small body population at the very outer edge of the solar system, now known as the Oort Cloud, has also been imagined for a long time since the 1950s. In 1950, J. H. Oort advocated the idea that our solar system is surrounded by a distant cloud of comet stuff (Oort, 1950). Although the existence of the Oort Cloud was proposed almost at the same time as that of the Kuiper Belt (Edgeworth, 1949; Kuiper, 1951), there is a significant difference between them: many KBOs have already been discovered and confirmed, while none of the Oort Cloud objects has yet been directly detected. This is mainly because of their very large distance from the Earth and the Sun. However, we can anticipate their existence by the flux and distribution of long-period comets coming all the way to the inner solar system. Comet clouds such as the Oort cloud are expected to exist not only in our solar system but around other stars. Consult Dones *et al.* (2004) for a comprehensive review of the cometary dynamics and the Oort Cloud formation.

The pioneering work in comet cloud dynamics has been carried out by M.J. Duncan, T. Quinn, and S. Tremaine (e.g. Duncan *et al.*, 1987), based on a dynamical model of galactic tidal fields (e.g. Heisler and Tremaine, 1986; Heisler *et al.*, 1987). Duncan *et al.* (1987) simulated the formation of the comet cloud and its subsequent evolution over the age of the solar system. Their simulations showed that the formation of the current comet cloud was driven mainly by an interaction between planetary perturbations and torquing due to galactic tides. The inner edge of the cloud is estimated at about 3000 AU, the radius where the timescales for the two processes are comparable. Duncan *et al.* (1987) also suggested that the flux of comets into the inner solar system initiated by the close passage of a star may be up to 20 times higher than the steady state rate. See also Duncan *et al.* (1988), Quinn *et al.* (1990), or Hogg *et al.* (1991) for other publications about comet cloud dynamics by S. Tremaine and his colleagues.

The dynamical source of long-period comets is often assumed to be around the Uranus–Neptune zone, since Jupiter and Saturn are considered to have ejected most of their icy planetesimals to interstellar space. The expected fraction of planetesimals ejected from each giant planet zone to the Oort Cloud shows that the relative efficiency of Uranus and Neptune in placing objects in the Oort Cloud far exceeded that of Jupiter and Saturn, by factors up to 20 or more (cf. Fernández and Ip, 1981; Duncan *et al.*, 1987; Fernández, 1997). On the other hand, Higuchi *et al.* (2006) investigated the first dynamical stage of comet cloud formation, the scattering of planetesimals by a planet. Applying their results to the solar system Oort Cloud, they found that Jupiter is the planet most responsible for producing candidate elements of the Oort Cloud, as long as the inclination of planetesimals is constant or proportional to the reduced Hill radius of each planet.

Another group that has been working on dynamical origin of the Oort Cloud is L. Dones, H. Levison, P. R. Weissman and their colleagues. In their series of numerical integrations (e.g. Dones *et al.*, 1998; Levison *et al.*, 1999; Dones *et al.*, 2000) they calculated the orbits of thousands of test particles from initially low-inclination and low-eccentricity orbits with semi-major axes of 4–40 AU over 4 billion years. They assumed that the long-period comets reached the Oort Cloud through planetary perturbations and galactic tides, and tried to determine the birthplace of the Oort Cloud comets. Levison *et al.* (2001b) studied the origin of Halley-type comets using the orbital integration of many test particles initially entering the planetary system from the Oort Cloud. They found that an isotropically distributed Oort Cloud does not reproduce the observed orbital element distribution of the Halley-type comets. Levison *et al.* (2002) calculated the number of dormant, nearly isotropic Oort Cloud comets using various orbital distribution models. According to them, we should have discovered $\sim$ 100 times more dormant, nearly isotropic comets than are actually seen under the assumption that comets are never destroyed. This result indicates that the majority of comets must physically disrupt as they evolve inward from the Oort Cloud. The Oort Cloud is suspected, incidentally, to be seamlessly connected to the inner part of the TNO region: the scattered disk. See Levison and Duncan (1997), Duncan and Levison (1997), Levison *et al.* (2001b), or Levison *et al.* (2006) for the relation between the Oort Cloud dynamics and the scattered disk.

The question on the dynamical source of the Oort Cloud objects (or long-period comets) will be open until we have clearly known their accurate spatial distribution. Overall, the dynamical study of the Oort Cloud is not yet quite strongly constrained by observations; we have not seen any of the Oort Cloud object at their home region of several thousand AU away. Although the direct detection of Oort Cloud objects in their home region of 3000 AU or further is not easy, it is not impossible either, using star-object occultation event sensing (cf. Kaplan *et al.*, 2003) or spacecraft missions. Once the Oort Cloud objects are directly detected, a significant amount of information on the dynamical characteristics of the small body population in the outermost area of the solar system will be delivered to us. Also, accurate observational data obtained from the ground-based and satellite astrometry provide us with more and more definitive information about the structure of our galaxy and the possible history of stellar passages near the edge of our solar system (e.g. García-Sánchez *et al.*, 1999, 2001). This will tell us more about dynamical history and evolution process of the Oort Cloud.

4.5. *Planetary rotation*

Dynamics of planetary rotation, especially those of the Earth, has been examined in a wide range of research fields, according to the timescales considered. The timescales range from an hour or a day for the nutation with small amplitude, to billions of years for the secular change of the Earth's spin state due to the tidal evolution of the Earth–Moon system.

4.5.1. *Nutation and precession of the Earth*

Research on the Earth's rotational motion with short timescale (days to months) is renowned for its high accuracy. A hundred years have passed since H. Kimura discovered the so-called z-term in the Earth's latitude variations (Kimura, 1902). The z-term was finally explained as an error of the coefficient of the semi-annual nutation term (Wako, 1970). There is a long history for this solution. In the ILS (International Latitude Service) network, before 1955, two groups of stars (six pairs of star in each group in two hours) had been observed every night at each of the latitude stations. The observational instrument was the VZT (Visual Zenith Telescope). One group of stars was replaced by another group in the next month. One year later, the first group of stars came back, and the cycle was closed. This chain method was to equalize positional errors of the stars in the catalogue. Kimura somehow noticed that the two-group observations are not enough to separate annual meteorological variation and semi-annual nutation. In the 1938 IAU (International Astronomical Union) general assembly, Kimura proposed, via Y. Hagihara, an observation program consisting of three groups of stars per night where one group of stars is replaced by another every month. Though this proposal was not adopted by the IAU in 1938, the ILS started the three-group observations in 1955. Then, soon the z-term turned out to represent the difference of the rigid Earth model and the non-rigid Earth model. Wako's result gave substantial support to the theory of the Earth rotation with a liquid core and an inner solid core developed by Jeffreys and Vicente (1957) and others. From the age of Kimura until the 1970s, this kind of research on the Earth rotation was done using latitude and polar motion observations by way of positional astronomy. Recently, these classical methods have given way to new space technology with very high accuracy, such as SLR (Satellite Laser Ranging), VLBI (Very Long Baseline Interferometry), and GPS (Global Positioning System). Now the accuracy of these observations is extremely high, requiring of the corresponding theories the highest accuracy possible.

As for the nutation of the rigid Earth, which has the shortest timescales among these sorts of phenomena, H. Kinoshita has made a significant contribution. The IAU (International Astronomical Union) had officially adopted the theory of E. M. Woolard (Woolard, 1953) as a standard model of the Earth nutation. However, Woolard's theory was no longer sufficiently accurate to compare with observations having very high accuracy using VLBI and other methods in the 1970s. Woolard's theory also had the defect of calculating the nutation of the instantaneous rotational axis of the Earth, not of the axis of the Earth's shape. Kinoshita (1977a) had brought a Hamiltonian dynamics into the field of the rigid body nutation of the Earth. He recalculated the nutation in terms of the axis of shape, the axis of instantaneous rotation, and the axis of angular momentum of the rigid Earth. The number of periodic components calculated in Kinoshita (1977a) is 106, and the order of error is better than 10^{-4} arc second. His theory was extended in Kinoshita and Souchay (1990) as well as Souchay and Kinoshita (1991).

At present, the difference between theoretical and observational values of the Earth nutation is recognized as being caused by the non-rigid components of the Earth's interior. T. Sasao had made a large and significant contribution to this field with a very accurate nutation theory encompassing the fluid dynamics of the Earth's core (Sasao *et al.*, 1980); the theory is succeeded by Shirai and Fukushima (2000, 2001). J. Getino is trying to incorporate the non-rigidity of the Earth rotation in Hamiltonian formalism (Getino *et al.*, 2001).

4.5.2. *The Earth–Moon system*

When the timescale of the variation in the Earth rotation becomes as long as 10^8 to 10^9 years, the dynamics inevitably involves the tidal evolution of the Earth–Moon system (e.g. Goldreich, 1966; Mignard, 1982). The consequence of secular changes in the Earth–Moon system is recorded in the geological and fossil record, such as in the striped bands on shell fish (e.g. Williams, 1990; Ito *et al.*, 1993). As is widely accepted now, the Moon was formed by a giant impact of a Mars-sized protoplanet on the proto-Earth (e.g. Hartmann *et al.*, 1986; Lee *et al.*, 1997a; Ida *et al.*, 1997; Canup and Righter, 2000; Kokubo *et al.*, 2000). Since then, the rotational rate of the Earth has been decreasing and the Moon has been moving gradually away from the Earth. It will soon be possible to numerically calculate the evolution of Earth–Moon dynamics fully including the interaction between the Earth's rotation and the Moon's orbital motion for billions of years. Some such trials have already been done by J. Touma and J. Wisdom using a sophisticated symplectic integrator (Touma and Wisdom, 1994, 1998, 2001). For this purpose, we need more accurate information on the time variation of ocean-continent distribution on the Earth as well as that on the Earth's and Moon's interior structure so that we can make a precise dynamical model of the lunar tidal torque.

As for relatively short timescales up to $O(10^3)$ years, we have to depend on ancient historical records concerning astronomical phenomena: solar and lunar eclipses, occultations of planets/stars by the Moon. Here the research on the Earth rotation becomes an interdisciplinary area involving the study of the ancient history of mankind (e.g. Stephenson, 1997; Tanikawa and Sôma, 2001). Most of these data come from Chinese chronicles and Babylonian astronomical diaries written on clay tablets. An accurate determination of $\Delta T = \mathrm{TT} - \mathrm{UT}$ has just recently begun (e.g. Stephenson and Morrison, 1995; Stephenson, 1997). Japanese and Chinese contributions to this field are highly anticipated (e.g. Han, 1997; Tanikawa and Sôma, 2001, 2004a,b; Kawabata *et al.*, 2002, 2004; Sôma *et al.*, 2004).

When the timescale becomes longer, up to several 10^4 to 10^5 years, what we can rely on is geological records such as the fossils of plankton in ocean drilling cores, data in ice drilling cores from the polar regions, or the annual rings of trees. These geological data constitute the fundamental materials underpinning a theory of long-term solar insolation radiation called the "Milankovitch cycles"; these cycles are due to the precession of the Earth's spin axis combined with the secular change in the Earth's orbital elements (cf. Milankovitch, 1920, 1930, 1941; Sharaf and Budnikova, 1969a,b; Berger, 1976; Berger *et al.*, 1984; Berger, 1988, 1989; Paillard, 2001). Now the Milankovitch cycles are being considered for bodies other than the Earth. The polar caps of Mars might retain some evidence of its climate change through the severe variation in its obliquity, which drives some theoretical work based on data obtained by planet exploring spacecrafts (e.g. Ward, 1974; Ward *et al.*, 1974; Laskar *et al.*, 2002; Head *et al.*, 2003).

4.5.3. *Evolution of planetary rotations*

Understanding evolution of planetary spins is an interesting but formidable task for solar system dynamics. Since the timescale of rotational motion is generally much shorter than that of orbital motion, it is not easy to investigate rotational motion as accurately and for such long terms as we do orbital dynamics over the timespan of solar system evolution. Also in the long-term evolution of planetary rotational and orbital motion, we have to consider the effect of their complicated coupling, so called "spin-orbital coupling" (e.g. Goldreich and Peale, 1966; Tomasella *et al.*, 1997; Murray and Dermott, 1999; Ward and Canup, 2006). Partly due to these difficulties, quite a few interesting problems still remain open in this line of research. For example, the origin of the retrograde rotation of Venus, the origin of the synchronization of Mercury's and the Moon's rotational motions with their orbital motions, and the origin of the inclined rotational axis of Uranus. Readers can consult discussions on the origin and evolution of planetary spin in the context of planet formation, for example, in Tanikawa *et al.* (1991), Ohtsuki and Ida (1998), or Agnor *et al.* (1999). Agnor *et al.* (1999) determined the spin angular momentum states of the growing planets by summing the contributions from each collisional encounter. Their results showed that the spin angular momentum states of the final planets are generally the result of contributions made by the last few large impacts. One of the latest treatment of the evolution of planetary spin is presented as a series of work by A. C. M. Correia (Correia and Laskar, 2001, 2003; Correia *et al.*, 2003) that presented a detailed formulation of spin axis dynamics of Venus with dissipation.

One additional complication to this research field is the fact that planetary rotations are considered strongly chaotic (cf. Laskar and Robutel, 1993; Laskar *et al.*, 1993a,b), and the detailed structure of the chaos is not yet well known. In other words, however, this fact means that the planetary rotation study can create a series of good example problems for the research of general dynamical systems.

Planetary rotational dynamics is quite relevant to research on extrasolar planetary systems in terms of the question of the possible existence of habitable planets. It is generally agreed that a habitable planet would be in a circumstellar “habitable zone,” on a nearly circular and planar orbit. The obliquity of the planet should not be so large, so as to avoid an extreme change of seasons (e.g. Williams and Kasting, 1997; Williams *et al.*, 1997). Condition for long-term dynamical stability and instability of planetary obliquity and its relationship to planetary climate is one of the key ingredients of the study of habitability in the future extrasolar planet research (e.g. Atobe *et al.*, 2004; Abe *et al.*, 2005; Atobe and Ida, 2007).

5. Numerical methods

Gordon E. Moore, one of the founders of Intel Corporation, made his famous observation in 1965, four years after the first planar integrated circuit was invented. It was called “Moore’s Law”. Moore (1965) observed an exponential growth in the number of transistors per integrated circuit and predicted that this trend would continue. As he predicted, the development of general-purpose microprocessors has continued, roughly doubling the number of transistors every 1.5 years for the past 40 years. For example, the Intel 80386 processor introduced in 1985 had only 275,000 transistors per chip, whereas the Pentium 4 processor in 2000 had about 42,000,000 transistors. This trend is expected to continue until around 2020 when atomic level accuracy will be required to produce integrated circuits. Needless to say, celestial mechanics also owes a great deal of its progress to the progress of this digital technology.

Averaged calculation speed of digital computers has increased nearly a hundred times in the past decade. In addition to the development of integrated circuits described by Moore’s Law, the architecture of computers has made great progress in recent years; from simple scalar computers through vector computers, to massively-parallel computers. However, the situation may change from this point on. The cost of producing new LSIs (large-scale integrated circuits) is getting higher, with much longer times involved. Designing a large LSI is no longer as easy a task as before. These difficulties make us hesitate to develop new LSIs, leading to higher production cost of LSIs. Then, the high cost makes the difficulties of designing new LSIs even greater: This is a bad positive feedback. Vector computers are almost dying out because of their low performance/cost ratio as well as because of their small market. Very few computer makers manufacture vector processors now, and it is just a matter of time before vector computers just disappear like dinosaurs.

Under this situation, the future of high-performance computing in celestial mechanics will be multidirectional; entailing sophisticated algorithms to accurately and efficiently solve equations of motion of celestial bodies, massively-parallel computers with relatively small cost, and special-purpose computers such as GRAPE. Having this background in mind, we describe in this section some points in terms of the development of numerical methods in celestial mechanics: some issues on regularization, recent progress in symplectic integrators, the special-purpose GRAPE supercomputer, and several related topics to this field. We are quite aware that many interesting and important aspects in this field are missing from this section due to authors’ inability. For example, we should have mentioned a lot more about symmetric integrators and their application to celestial mechanics as well as those of geometric integrators in a broader sense. Various techniques to deal with a huge number of particles that interact with each other through gravitational force, such as the tree method, are not described. Also, perhaps a brief summary of general development of computer hardware that is related to celestial mechanics should have been done. However, we hope that readers are able to get some clues about these issues from the descriptions and the references cited in this section.

5.1. *Regularization*

One of the most difficult tasks in numerical integration in celestial mechanics is the treatment of collisions. This is not only a numerical but also a theoretical problem. Regularization is a method used to deal with dynamical collisions numerically as well as analytically.

The restricted three-body problem admits binary collisions. Sundman (1912) has shown that the singularity of the equations of motion corresponding to a binary collision is not essential, but can be removed via a change of variables. This process is called a regularization. Regularization of a binary collision has been carried out by Thiele (1896), Levi-Civita (1906), Burrau (1906), and Birkhoff (1915). Thiele’s (1896) work concerned an equal-mass binary system, later extended to arbitrary mass ratio by Burrau (1906). It is to be noted that these regularizations are all for planar problems. Levi-Civita (1906) provided planes around the singularity. Whenever an orbit moves from one plane to another, it experiences a collision. Birkhoff (1915) devised a change of variables so that a binary collision either at the first or the second primary can be treated through a single formula. Thiele (1896) and Burrau (1906) transformed the variables to hyperbolic–elliptic coordinates. The Lemaître transformation (Lemaître, 1952) looks like that of Birkhoff, but the correspondence of old and new variables is 1 to 2 in the latter, whereas that is 1 to 4 in the former. Based on these pioneering studies, Kustaanheimo and Stiefel (1965) have succeeded in regularizing the three-dimensional problem, embedding three variables in the four-dimensional space. This is the celebrated Kustaanheimo–Stiefel (K–S) regularization (or the K–S transformation). See also Section 5.2 for incorporation of the K–S regularization into symplectic integrators by S. Mikkola [4].

Currently, the most significant and promising flow of regularization research goes on around the K–S transformation. For example, Arakida and Fukushima (2000, 2001) confirmed that the positional error of a perturbed two-body problem expressed in the K–S variables is proportional to the fictitious time s, which is the independent variable in the K-S transformation. This property does not depend on the type of perturbation, on the integrator used, or on the initial conditions, including the nominal eccentricity. The error growth of the physical time evolution and the Kepler energy is proportional to s when the perturbed harmonic oscillator part of the equation of motion is integrated by a time-symmetric integration formula, such as the leapfrog or the symmetric multistep method. The error growth is proportional to s^2 when using traditional integrators, such as the Runge-Kutta, Adams, Störmer, and extrapolation methods. Also, it turned out that the K–S regularization avoids the step size resonance or instability of the symmetric multistep method that appears in the non-regularized cases. Therefore, the K–S regularized equations of motion are found quite useful for investigating the long-term behavior of perturbed two-body problems, namely, those used for studying the dynamics of comets, minor planets, the Moon, and other natural and artificial satellites.

As applications of the K–S regularization, there are several methods to practically regularize triple collision. The first one is due to Aarseth and Zare (1974). Their idea is to introduce two K–S regularizations to two of the three edges of the triangle formed with three particles. The second one is the so-called "chain method" devised by Mikkola and Aarseth (1990, 1993). In this method, particles form a chain. To make a chain, we start from a closest pair of particles and form a chain made of a vector that connects the pair. Then, a particle closest to either end of the first chain is added to form a new chain. Next, the fourth particle is selected as the closet particle to the third one. This procedure is continued until all the particles are exhausted. After the completion of the chain, the K–S regularization is applied to each pair of the chain. The third example is the global three-body regularization that was devised by Heggie (1974). Heggie adopted a time transformation $dt = r_1 r_2 r_3 d\tau$ where r_i $(i = 1, 2, 3)$ are the mutual distances of particles. The new Hamiltonian Γ becomes $\Gamma = r_1 r_2 r_3 (H - E_0)$ where H is the original Hamiltonian and E_0 is its numerical value. Then the equations of motion are regular for collisions between any pair of particles. This method is accurate in the case of triple close approach though it is rather time-consuming. Other characteristics of this method are that the switching of the closest pairs is not necessary, and that it can in principle be applied to the N-body problem for any N. For subsequent developments, readers may consult papers by S. J. Aarseth and S. Mikkola (e.g. Aarseth, 1985, 1988; Mikkola and Aarseth, 1998, 2002), a review paper (Aarseth, 1999), and a textbook (Aarseth, 2003).

Quite recently, Fukushima (2007) announced that he found a new scheme to regularize a three-dimensional two-body problem under perturbations. His method is a combination of Sundman's time transformation and Levi-Civita's spatial coordinate transformation applied to the two-dimensional components of the position and velocity vectors in the osculating orbital plane. The equations of motion using his new variables have no singularity even when the mutual distance is extremely small. Hence the new variables are suitable to deal with close encounters. The number of dependent variables in the new scheme becomes eight, which is significantly smaller than the existing schemes to avoid close encounters: for example, the number is smaller by two compared with the K–S regularization.

5.2. *Symplectic integrators*

Since many of the systems that celestial mechanics is concerned with are Hamiltonian systems or their proximity, numerical integration schemes designed specifically to maintain the Hamiltonian structure are evidently desirable and promising. Symplectic integrators are exactly the methods that satisfy such requirements. In general, symplectic integrators conserve the total energy of a system quite well, preventing artificial damping or excitation due to the accumulation of local truncation errors (e.g. Yoshida, 1990b; Gladman *et al.*, 1991; Kinoshita *et al.*, 1991; Yoshida, 1993; Sanz-Serna and Calvo, 1994; Morbidelli, 2002). The mathematical background of symplectic integrators is based on Lie algebra. Its application to physics is described, for example, in Varadarajan (1974), Dragt and Finn (1976), Ruth (1983), or Yoshida (2001). Yoshida (1990b) first noticed that a higher-order symplectic scheme can be composed of several second-order schemes, and derived some practical pairs of coefficients for higher-order integrators. Yoshida also proved that a class of explicit symplectic integrators rigorously preserve the total angular momentum within the range of round-off errors when used for gravitational N-body problems (Yoshida, 1990a).

In practice, general-purpose symplectic integrators

[4] In all regularizations described in this paragraph, time t is transformed in the form $dt = r d\tau$ near collisions where τ is the new time variable and r is the mutual distance between colliding bodies. Usually, the solution passes through singularity within a finite time. For comparison, we should notice that the McGehee transformation (McGehee, 1974) adopts the form $dt = r^{3/2} d\tau$ so that the solution does not pass through singularity: it takes an infinite time to arrive at singularity. See also Section 3.1 for the McGehee transformation (or the McGehee variables).

tend to be of low order, mainly because of their complexity in higher orders. Therefore they are not suitable for long-term numerical simulations. However, the so-called Wisdom-Holman type symplectic map ("WH map") devised by Wisdom and Holman (1991), which is specialized for slightly perturbed motion like solar system planetary motion, is more accurate by orders of magnitude than the general-purpose symplectic method even when it is of low order. The principle of the WH map is to split the Hamiltonian into an unperturbed part (for example, Kepler motion when we trace planetary motion) and a perturbation part (for example, mutual gravitational perturbation). In each step of the integration, the system is first drifted forward in time according to unperturbed motion, and then a kick in momentum is applied that is derived from the perturbation part of the Hamiltonian. The first step is analytic because the Hamiltonian is unperturbed and integrable. When this part deals with Keplerian drift, a set of canonical coordinates, such as Jacobi coordinates (e.g. Plummer, 1960), is used to keep the integration scheme symplectic. Using this scheme, the WH map enhances the speed of long-term numerical integrations of planetary motion by orders of magnitude. Wisdom and Holman (1992) investigated the numerical stability of the WH map, and discussed the existence of numerical instability within the map. Note that when using the WH map for planetary dynamics, we have to solve the Kepler equation every time in the drift phase. As for the fast solving methods of the Kepler problem, we may find it helpful to consult a series of papers by T. Fukushima (1996a, 1997a,b,d, 1998).

After the monumental work by Wisdom and Holman (1991), a lot of enhancements have been added to the WH map. Based on the idea of the WH map, Saha and Tremaine (1992) discovered a special start-up procedure to reduce the truncation error of angle variables, called the "warm start", exploiting one of the characteristics of Hamiltonian systems—the existence of adiabatic invariants. They also discovered a symplectic scheme with individual stepsizes, dividing the total Hamiltonian into each planet's Kepler and perturbation parts (Saha and Tremaine, 1994). A technique related to the warm start was devised by Wisdom *et al.* (1996) under the name of the "symplectic corrector". The symplectic corrector exploits the fact that the values of symplectically integrated systems (i.e. coordinates and momenta) are obtained by canonical transformations of the variables in real systems. See also Wisdom (2006) for application of the symplectic corrector to the N-body system described by canonical heliocentric coordinates. Michel and Valsecchi (1997) discussed in detail the efficiency of the WH map compared with traditional integration schemes such as the Bulirsh-Stoer extrapolation method. For higher order symplectic methods for nearly integrable Hamiltonian systems, consult recent publications such as Chambers and Murison (2000) or Laskar and Robutel (2001).

Now one of the biggest issues concerning symplectic integrators is how to handle close encounters between particles where the variation of stepsize is inevitable, because symplectic integrators are inherently not well matched with variable stepsize schemes (Yoshida, 1993). Based on some preceding studies (Skeel and Biesiadecki, 1994; Lee *et al.*, 1997b), Duncan *et al.* (1998) has introduced a method to solve the problem with the aid of "democratic heliocentric coordinates", also known as canonical heliocentric coordinates that use heliocentric positions and barycentric momenta (cf. Charlier, 1902). Their method enables us to handle close encounters among particles by splitting gravitational potentials recursively. The method maintains computational speed equivalent to that of the generic WH map when no close encounter occurs. When a close encounter happens, the scheme automatically divides the stepsize and alleviate numerical errors. Duncan *et al.* (1998) named the scheme SyMBA, or "symplectic massive body algorithm". Their program package is named SWIFT, and is available on their ftp site. One of the major disadvantages of using SyMBA is that it cannot take care of highly eccentric orbits, i.e. close encounters with central mass. The other problem is the complexity of its implementation.

Chambers (1999) originally devised a method similar to SyMBA but with a slightly different idea. He noticed that the drift part of the symplectic scheme does not need to be calculated analytically, but can be integrated numerically using a very accurate method such as the extrapolation scheme. When a close encounter happens, his scheme moves the relevant terms in the perturbation Hamiltonian H_{pert} to the Keplerian Hamiltonian H_{kep}, and keeps the contrast of the order of magnitude of Hamiltonians, $H_{\mathrm{kep}} \gg H_{\mathrm{pert}}$. This hybrid scheme enables us to calculate the orbital motion in the system where close encounters frequently occur, with less programming cost than SyMBA. Chambers *et al.* (2002) later extended this hybrid method to a scheme specialized for orbital motions in and around binary stars, choosing suitably canonical variables. This method is available for integrating planetesimals' and protoplanets' orbital evolution near binary star systems (e.g. Quintana *et al.*, 2002; Quintana and Lissauer, 2006; Thébault *et al.*, 2004).

Incorporating the above idea of Chambers *et al.* (2002) into SyMBA, Levison and Duncan (2000) has developed a modified version of SyMBA that can take account of close encounters with the central mass, i.e. highly eccentric orbits. Their method is available for systems where close encounters between particles as well as between a particle and the central mass frequently occur. When neither a close encounter happens nor the orbits of particles are so eccentric, the modified SyMBA performs as fast as the generic WH map does. Except for the complexity of its implementation, the modified SyMBA is one of the ultimate schemes in symplectic integrators that are suitable for planetary dynamics in planet formation stage.

In a practical use of symplectic integrators for highly eccentric orbital motions, a series of studies on regularized symplectic integrators by Mikkola and collaborators seems quite promising (Mikkola, 1997, 1999; Mikkola and Innanen, 1999; Mikkola and Tanikawa, 1999a,b; Mikkola and Palmer, 2000). They have brought the K–S regularization into the WH map, keeping its symplectic structure intact, though regularization had long been considered incompatible with symplectic integration. Their basic idea is to take the logarithm of the regularized Hamiltonian. In order to regularize the equations of motion, one multiplies distance r to the original Hamiltonian, which is equivalent to division of the Hamiltonian by the dominant potential. This means that the new Hamiltonian is not separable, i.e. coordinates and momenta are mixed, even when the original Hamiltonian is separable. Taking the logarithm of the new Hamiltonian, we can separate variables and exploit the leapfrog procedure for symplectic integration.

Mikkola's (1997) method is not only accurate when adapted for highly eccentric orbits, but also quite fast, since there is no need to solve Kepler's equation which needs certain iterative procedures involving transcendental functions. Compared with similar previous work, such as Levison and Duncan's (1994) "RMVS" (regularized mixed variable symplectic integrator) by changing stepsizes automatically and by force-center switching, Mikkola's method is more efficient, more accurate, and easier to implement. Breiter (1999) also found a regularized method using K–S variables, which he insists is a similar but complementary approach to the work of Mikkola. See also Preto and Tremaine (1999) for a class of symplectic integrators with adaptive stepsize for separable Hamiltonian systems.

From this point of view, Rauch and Holman's (1999) intensive review is quite worth reading. They have tested many variants of the WH map and measured their efficiency. They have concluded that an enhanced version of a potential splitting method such as SyMBA, incorporating time regularization, force-center switching, and an improved kernel function, may be quite efficient when particles are subject to both high eccentricities and close encounters. To integrate eccentric and nearly Keplerian orbits without close encounters, the time-transformed WH map by Mikkola is clearly the most efficient and stable.

Symplectic integrators can be extended to include dissipative mechanisms. Malhotra (1994) devised such a method to integrate the planetary orbital motion dragged by gas, adding a simple modification to the generic WH map so that it can include velocity-dependent forces. Mikkola (1998) discussed the inclusion of non-canonical perturbations in symplectic integration schemes using the δ-function formalism. J. Touma's series of work (Touma and Wisdom, 1994, 1998, 2001) demonstrates the feasibility of exploiting symplectic integrators to calculate systems with long-term tidal dampings, such as the Earth–Moon system and the core-mantle coupling of the Earth's interior.

The research devoted to the enhancement and improvement of symplectic integrators, especially those of the WH map, is still going on. For example, Liao (1997) gave schemes for a low-order mixed symplectic integrator for an inseparable, but nearly integrable Hamiltonian system. Although Liao's schemes are implicit, they have a faster convergence rate of iterative solutions than ordinary implicit integrators do. Fukushima (2001) proposed and examined three kinds of approach to reduce the accumulation of round-off errors in symplectic integrators: the reduction of the number of summations in the implementation of symplectic algorithms, the use of a double-length routine library in the main summation procedure, and the full use of the double-length routine library in the entire procedure of symplectic integration (note that the last two methods will not be applicable to the WH map unless double-length routines to evaluate trigonometric functions are provided). Beust (2003) presented a new symplectic integrator for massive bodies that permits the numerical integrations of the dynamics of hierarchical stellar systems of any size and shape, provided that the hierarchical structure of the systems is preserved along the integration. Chambers (2003) noticed that the previously known symplectic methods that are higher than second order include some substeps that travel backward, compared with the main integration. To compensate for this, some substeps in these methods have large coefficients, which could produce large error terms and reduce the efficiency of high-order symplectic algorithms. Chambers introduced complex coefficients in higher order symplectic integrators from the solution of the constraint equations for the substeps of symplectic algorithms. He showed that complex integrators with leading error terms that have strictly imaginary coefficients effectively behave as if they are one order higher than expected.

Symplectic integrators can be improved in terms of force calculation. As is well known, the amount of direct computation of the distances between each pair of gravitationally interacting N particles grows as $O(N^2)$. To overcome this difficulty, a variety of fast (with effort growing more slowly than $O(N^2)$) but approximate force calculation methods such as those used in the tree code or fast multipole methods have been developed. Wiegert *et al.* (2004) examined several of these algorithms together with symplectic integrators, and compared their speed in very detail. Also, combination of the special-purpose computer GRAPE (Section 5.4) with symplectic integrators, especially those of SyMBA-type, might have a possibility to greatly enhance the computing speed of gravitational N-body interactions in solar system dynamics.

5.3. *Other promising numerical techniques*

There will be two major streams of numerical integration schemes used in celestial mechanics. One of them is the symplectic integrators that we have already mentioned in the preceding section, and the other is symmetric integrators (cf. Hairer *et al.*, 1993, 2002). Symplectic integrators will be developed in various ways specialized for each dynamical problem, such as versions for binary systems, three-body systems with regularization, systems with frequent close encounters, and so on. Symmetric methods have a superiority over the symplectic integrator in that it is easy to construct variable stepsize schemes. Among many kinds of symmetric integrators, fourth-order time-symmetric methods based on the Hermite integrator seem to be full of promise, especially for problems of planetary accretion or stellar dynamics where frequent close encounters require variability in stepsizes (e.g. Makino, 1991b; Makino and Aarseth, 1992; Kokubo *et al.*, 1998; Kokubo and Makino, 2004). The Hermite scheme is also a good match with the special-purpose GRAPE computer for the gravitational N-body problem (Makino, 1991a).

While the Hermite-type integrators are one-step methods, higher-order multistep symmetric integrators are also widely used in planetary dynamics where a very high accuracy is required. Following a pioneering work by Lambert and Watson (1976), Quinlan and Tremaine (1990) proposed an efficient and useful 12th-order scheme, which is now a standard of the higher-order linear multistep symmetric method. Later, Evans and Tremaine (1999) as well as Fukushima (1999b) investigated the numerical instability of the multistep symmetric method independently, and proposed several improved schemes. Though it is not easy to implement a variable stepsize scheme in multistep symmetric methods, they are quite suitable for planetary dynamics with regular and smooth motion without close encounters between planets. One of the current major problems of multistep symmetric methods is that, they are subject to intrinsic stepsize resonances and instabilities which were first recognized by Alar Toomre (cf. Quinlan, 1999; Fukushima, 1999b). Hence, conventional Störmer–Cowell methods (cf. Hairer *et al.*, 1993) may still be more widely used than multistep symmetric methods.

Dynamical models that we deal with in the solar system celestial mechanics is often nearly integrable, such as the quasi-Keplerian motion. This characteristic has been employed not only by the Wisdom–Holman symplectic map, but using traditional methods such as the Encke method (Encke, 1854; Fukushima, 1996b). Recent works by T. Fukushima, collectively called "the method of manifold correction", also exploits this quasi-Keplerian character of solar system dynamics. Fukushima (2003c) proposed a new approach to numerically integrate quasi-Keplerian orbits. His method integrates the time evolution of the Kepler energy and the usual equations of motion simultaneously. It directly adjusts the integrated position and velocity by a spatial scale transformation in order to satisfy the Kepler energy relation rigorously at every integration step. By adding the Laplace integral as the second auxiliary quantity to this method, Fukushima (2003a) extended his scaling method to integrate quasi-Keplerian orbits in order to suppress the growth of integration errors, not only in the semimajor axis, but also in the other orbital elements, especially in the eccentricity and in the longitude of pericenter. Then again, by adding the orbital angular momentum vector as another auxiliary quantity to be integrated, Fukushima (2003b) extended his scaling methods for quasi-Keplerian orbits in order to suppress the growth of integration errors in the inclination and the longitude of the ascending node. His new approaches provide a fast and high-precision device to simulate the orbital motions in solar system dynamics at a negligible increase in computational cost. Fukushima's manifold correction method, combined with the Kustaanheimo–Stiefel regularization, is now creating a new, substantially large flow in this line of research (Fukushima, 2004a,b,c,d,e,f, 2005a,b,c,d).

Another important issue concerning numerical integration techniques is parallel computing. Since many problems that are dealt with in celestial mechanics do not involve very large number of bodies such as the solar system planetary motion, efficient parallelization of computing is not easy. However, recent investigation suggests a possibility for us to utilize a large parallel computer for a relatively small number of celestial bodies. For example, Ito and Fukushima (1997) developed a parallelized version of the extrapolation method to integrate general ordinary differential equations. A simple technique made its load balance among computer processors almost equal, nearly independent of how large amount of data (or how many particles) we use in our numerical model. Also, a method developed by Fukushima (Fukushima, 1997c,e, 1999a) expands an approximate solution of the equations of motion of celestial particles into Chebyshev polynomials, and approaches the true solution through the Picard iteration. Though the Picard–Chebyshev method is not yet being widely applied to real orbital problems (cf. Arakida, 2006), this method is expected to be quite suitable for systems with small perturbation, such as orbital and rotational motion in our solar system calculated with large parallel computers. Similar to the Picard–Chebyshev method, Saha and Tremaine (1997) have devised a new parallel computing algorithm for planetary orbital motion, especially suitable for scalar massive parallel computers. In addition, the near-future version of the special-purpose computer GRAPE, GRAPE–DR, will enable us to run a large number of small-scale gravitational N-body integrations all at once, which will practically be quite a massively parallelized calculation (see also Section 5.4).

Yet another important issue along this line of research can be the multiple precision arithmetic. Long-term numerical integration potentially and inevitably involves a large amount of round-off error. As mentioned before, several methods to alleviate the round-off error have been developed by Quinn and Tremaine (1990) and Fukushima (2001). Although their methods are algorithmically realized (i.e. by software), the hardwired implementation of multiple precision might now be possible for astronomical application. The implementation of hardwired multiple precision would also be quite influential on numerical experiments in theoretical studies, such as in helping to gain information on the detailed structure of various KAM tori, aiding in inspection of dependency of the Lyapunov exponents on arithmetic precision, and furthering many other problems in celestial mechanics and dynamical theory.

5.4. *GRAPE: A special-purpose computer for N-body dynamics*

As for numerical research in the fields of celestial mechanics and dynamical astronomy, we cannot avoid mentioning the enormous success of GRAPE (GRAvity piPE), a special-purpose supercomputer for gravitational N-body problems developed by a dedicated research group in Japan (e.g. Sugimoto *et al.*, 1990; Ebisuzaki *et al.*, 1993; Makino *et al.*, 1997; Makino and Taiji, 1998). GRAPE has created a revolutionary breakthrough in the area of gravitational N-body problems. The peak speed of its latest version, GRAPE–6, has a capacity of more than several tens of TFlops. The near-future goal of the GRAPE project is called GRAPE–DR, which will have a peak capacity of 2 PFlops or higher (Makino, 2005).

The essential feature of the GRAPE concept is the pipeline. On GRAPE, computation of the N-body problem is divided into two parts. The first part is performed on a host computer (such as a PC), and the other is on a special-purpose back-end processor (GRAPE). GRAPE calculates the gravitational force among particles through pipeline(s). During one timestep, the host computer performs $O(N)$ operations, while GRAPE can do $O(N^2)$ because of the devoted parallel pipeline. Applegate *et al.* (1986)'s "The Digital Orrery" also has a parallel architecture in which small computers are connected in a ring. Each processor takes care of one planet (particle). The Digital Orrery did not adopt a pipeline architecture because this architecture lacks the efficiency necessary for small N. Pipelines work more efficiently when N is sufficiently large.

The GRAPE project team started the development of GRAPE systems in 1989. The first hardware, GRAPE–1 was a single-pipeline system with low-accuracy force calculation (Ito *et al.*, 1990). It was a machine for demonstrating that theoretical astrophysicists could construct a practical hardware. It had a peak speed of 240 MFlops, and turned out to be useful for a wide range of astrophysical problems where high accuracy is not very important. GRAPE–2 was again a single-pipeline system, but with high accuracy, with a peak speed of 40 MFlops (Ito *et al.*, 1991). GRAPE–3 was the first multiple-pipeline system with a low-accuracy force calculation pipeline (Okumura *et al.*, 1993). The peak speed of GRAPE–3 was 380 MFlops/chip.

As a successor of the high-precision machine GRAPE–2, GRAPE–4 was developed (Makino *et al.*, 1997). The peak speed of the GRAPE–4 system is 1.08 TFlops, achieved by running 1692 pipeline LSIs in parallel, each providing 640 MFlops. The successor of the low-precision GRAPE–3 machine is GRAPE–5 (Kawai *et al.*, 2000). A GRAPE–5 board has a peak performance of 38.4 GFlops. The newest edition of the GRAPE system is GRAPE–6 (Makino *et al.*, 2003), the successor of the high-precision machine (GRAPE–4), and its single PCI card version, GRAPE–6A (Fukushige *et al.*, 2005). Similar to GRAPE–4, the primary application of GRAPE–6 is simulations of collisional systems, though it can also be used for collisionless systems. Various improvements have been done for GRAPE–6 from GRAPE–4, which ended up with a peak speed of 64 TFlops.

We must also mention that the GRAPE system has the capability of hardwired implementation of particular algorithms for astronomical applications. This kind of GRAPE is called PROGRAPE (Hamada *et al.*, 2000). Now PROGRAPE–1 (PROgrammable GRAPE–1), a programmable multi-purpose computer for many-body simulations, is available. The main difference between PROGRAPE–1 and traditional GRAPEs is that PROGRAPE–1 uses FPGA (Field Programmable Gate Array) chips as its processing elements, while the traditional GRAPEs rely on a hardwired pipeline processor specialized for gravitational interactions. GRAPE–7, the successor of the lower-precision machine (GRAPE–5) but now with FPGA, will also be available shortly (Kawai and Fukushige, 2006).

The GRAPE project is tremendously successful in the field of gravitational N-body dynamics, especially for the planet formation and collisional stellar dynamics. Hundreds of academic papers have been published using GRAPE computation results, and the number is ever increasing. See the GRAPE project's webpage for a more detail (http://www.astrogrape.org/ for general information, and http://grape-dr.adm.s.u-tokyo.ac.jp/ for the GRAPE–DR project).

6. Concluding Remarks

As we have seen, since the beginning of the twentieth century, celestial mechanics has pretty much diverged into many specialized fields that do not communicate much with each other. Despite the apparent resulting prosperity of celestial mechanics, we suspect that this diversity could produce its potential stagnation through a lack of communication between its various fields. Perhaps it might be useless to imagine how many people have read and fully understood the five texts that all have the phrase "celestial mechanics" in their titles[5]: Smart (1953) *Celestial Mechanics,* Siegel

[5] The fact that popular textbooks such as listed here are mainly written in English allows them to be widely distributed and read all over the world. However, we should not forget that there are also many good textbooks written in other languages. The speed of understanding is clearly fastest when readers obtain information in

and Moser (1971) *Lectures on Celestial Mechanics,* Taff (1985) *Celestial Mechanics: A Computational Guide for the Practitioner,* Brumberg (1995) *Analytical Techniques of Celestial Mechanics,* and Poincaré (1892) *Les Méthode Nouvelles de la Méchanique Céleste.*

At this point, we would like to ask ourselves a very primitive question to which we have not found an answer: What is celestial mechanics? What does celestial mechanics aim at? It seems that there is no general agreement on the answer to these questions. However, this situation may not be so strange: Compared with other fields in astronomy, the disagreement among celestial mechanists on the subject (or target) of celestial mechanics looks remarkably large, partly because this field has a much longer history than other fields do. This is what has made celestial mechanics branch out into so many and such detailed paths.

Astronomy has expanded by an inconceivable degree in the twentieth century. Researchers extended the frontier of astronomy to various electromagnetic wavelengths in the first half of the twentieth century, beginning from visible light, reaching radio, infrared, ultraviolet, X-ray, and γ-ray. Astronomy in the twentieth century was full of discoveries and astonishment, most of which were brought about using the methods of modern astrophysics. Celestial mechanics has rolled up and down in the midst of the killer waves of astrophysics. When we remind ourselves of the ages before recent discoveries in astrophysics, it is not wrong to say that celestial mechanics was the unique field of quantitative science that covered the entire world known to humankind: In the eighteenth century, the solar system was almost the universe itself. In the nineteenth century, there was no notion of extra galaxies, and our galaxy was the universe. Spectroscopy of stars started only at the end of the nineteenth century. No one knew of nuclear reactions, and astrophysical studies of extrasolar stars were just beginning. These facts imply that celestial mechanics was at the center of the astronomy of those ages. With those centuries behind us, now the world is complicated enough to make it virtually impossibly to grasp the appropriate position of celestial mechanics in the vast sea of astronomy.

As it is not difficult to imagine, we are not the first to suffer from this kind of confusion. Already in the early 1920s when Brown *et al.* (1922) reported the activities of American celestial mechanists, they were already not sure what the main purpose of celestial mechanics was. Their reports consisted of three parts: the first one was devoted to the solar system, the second one was to celestial mechanics as applied to the stars, and the third one was to the theory of the problem of three and more bodies. At the very beginning of their report, Brown wrote:

> "Celestial mechanics, broadly interpreted, is involved in practically all the astronomy of the present time. The limited meaning of the term now usually adopted refers only to those problems in which the law of gravitation plays the chief or only part, and more particularly to those which deal with motions of bodies about one another and with their rotations."

which obviously indicates that Brown felt the same perplexity as what we feel now when he tried to define celestial mechanics at that stage of the early twentieth century.

When we are involved with celestial mechanics, an important attitude we should take is to always keep an eye on the concrete, actual, real phenomena in the universe, even when the methodology we use is highly mathematical or abstract. We should keep in mind that celestial mechanics is aimed at better understanding the real structure of the real world, not at fiddling with complicated equations or manipulating high-speed computers as a hobby. With this in mind, we will not lose the loose ends of celestial mechanics in our effort to understand the behavior of this world in terms of dynamics, no matter how varied the phenomena we encounter are.

Before the end of this manuscript, we would like to propose three potential near-future projects that are tightly related to the issues that we have so far discussed for the further development of celestial mechanics in the twenty-first century:

(i) To build a computer chip at least with 10,000-digit arithmetic, in order to observe complicated chaotic behaviors in dynamical systems such as the Arnold diffusion. As an example, take discrete dynamical systems, i.e. maps. At the worst case, numerical errors can be doubled per one iteration. This means that numerical errors can be multiplied thousand times over ten iterations. In other words, we can iterate the map thirty thousand times before we totally lose accuracy if we have ten thousand digits. This will enable us to directly calculate topological entropy of general dynamical systems. We expect that million-digit arithmetic may be possible by the end of the twenty-first century.

(ii) Occultation observations by the Moon-orbiting satellite on polar orbit, in order to survey multiple stars in our galaxy (cf. Tanikawa and Mikami, 2000). This will provide us unbiased data of visual binaries within a relatively short time, which will be the fundamentals not only for star formation study but for the celestial mechanics, particularly the dynamical study

their mother language. In this sense, we much prefer the existence of good textbooks written in various languages. However, as expected, non-English textbooks have only a limited number of readers. We do not think this is desirable. Good textbooks should be read by as many readers as possible, regardless of the languages in which they are written. One way to overcome this dilemma is to write your textbook in your own language as well as in English if you are the author. It cannot be very difficult to translate your own book into English. Fortunately, textbooks for celestial mechanics are generally full of mathematical formulas, which means you likely have little to translate into English. This seems only possible through the voluntary effort of ungrudging scientists.

of binary formation in primordial star clusters.

(iii) To attempt space trigonometry with baseline of $O(100)$ AU using two pairs of spacecrafts: the first pair orbits near around the ecliptic, and the other orbits in a plane that is perpendicular to the ecliptic. The purpose of this project is to make a three-dimensional map of our galaxy with the spatial scale of 10^4 pc and with the accuracy of $0''.01$ or higher, which will be the basis of future stellar dynamics. We will be able to get proper motion data over 1–2 years from this mission. Three-dimensional position data together with velocity data (i.e. proper motion and radial velocity) will provide us basic information about the dynamics of our galaxy. The advantage of this project is that trigonometric parallaxes can be obtained at once: we do not need to wait half a year. A disadvantage is that since the spacecrafts move so slowly that the configuration of stars and spacecrafts does not change so quickly. This disadvantage can be avoided, however, by increasing the number of spacecrafts in a pair (a group) from two to four.

In 1974, M. W. Hirsch and S. Smale commented on Newton's gravitational equations of motion in their celebrated textbook (Hirsch and Smale, 1974, p. 289) thus:

> "... The flow obtained from this differential equation then determines how a state moves in time, or the life history of the n bodies once their positions and velocities are given. Although there is a vast literature of several centuries on these equations, no clear picture has emerged. In fact it is still not even clear what the basic questions are for this "problem"."

We think this sarcastic and rather cynical comment reflects the huge potential of this field that has not yet been fully examined. The need for celestial mechanics will be felt to an even greater degree as human-being launch more artificial satellites and spacecrafts, recalling that the age of modern celestial mechanics began with Sputnik in the 1950s. Human beings might need to protect the Earth from the potential impacts of near-Earth objects or space debris, and again the knowledge and experience of celestial mechanics will be required with urgency. The future of celestial mechanics is closely connected with engineering and space technology, employed both inside and outside our solar system. We are grateful to mother nature for affording us the opportunity to witness the dawn of celestial mechanics in the new century.

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