Initializing Relativistic Velocity Distribution Functions in Plasma Simulations

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Particle-in-cell (PIC) simulations and Monte-Carlo simulations are important research tools in modern astrophysics. In these numerical simulations, particle velocities are usually initialized by random variables with certain algorithms. For example, the Box–Muller algorithm [1] is popularly used to generate a Maxwell–Boltzmann distribution. It is summarized in Table 1 and described in many textbooks. One can also incorporate the bulk velocity, by considering an offset to the particle velocities.

Since relativistic plasma processes have been growing in importance in high-energy astrophysics, there is a strong demand for algorithms to deal with relativistic velocity distribution functions of plasmas. However, algorithms to generate relativistic velocity distributions are not well established. Here, "relativistic velocity distributions" contain both relativistically-hot distributions and/or relativistically-moving distributions.

Our recent article [2] summarize the state-of-art algorithms to deal with both of them. Going back to the original article by Russian mathematician Sobol [3], we review standard algorithms [4] to generate a relativistic Maxwellian (Jüttner–Synge distribution) at a rest frame. We further propose two rejection methods to deal with the spatial part of the Lorentz transformation of arbitrary distribution functions, which has never been discussed in previous literature. These results are summarized in Table 2. We note that our algorithms are drastically simpler than similar attempts in recent years.

We hope that these algorithms are useful in relativistic kinetic simulations in high-energy astrophysics.

Table 1: Box–Muller algorithm.

generate X_1, X_2, X_3, X_4, uniform on $(0, 1]$
$v_x \leftarrow \sqrt{-2\ln X_1}\sin(2\pi X_2) + V_0$
$v_y \leftarrow \sqrt{-2\ln X_1}\cos(2\pi X_2)$
$v_z \leftarrow \sqrt{-2\ln X_3}\sin(2\pi X_4)$
return v_x, v_y, v_z

Table 2: Sobol algorithm with the flipping method.

repeat
  generate X_1, X_2, X_3, X_4, uniform on $(0, 1]$
  $u \leftarrow -T \ln X_1 X_2 X_3$
  $\eta \leftarrow -T \ln X_1 X_2 X_3 X_4$
until $\eta^2 - u^2 > 1$.
generate X_5, X_6, X_7, uniform on $[0, 1]$
$u_x \leftarrow u\ (2X_5 - 1)$
$u_y \leftarrow 2u\sqrt{X_5(1-X_5)}\cos(2\pi X_6)$
$u_z \leftarrow 2u\sqrt{X_5(1-X_5)}\sin(2\pi X_6)$
if $(-\beta v_x > X_7)$, $u_x \leftarrow -u_x$
$u_x \leftarrow \Gamma(u_x + \beta\sqrt{1+u^2})$
return u_x, u_y, u_z

References

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[3] Sobol, I. M.: 1976, in Monte Carlo Methods in Computational Mathematics and Mathematical Physics, p.24 [in Russian].
[4] Pozdnyakov, L. A., Sobol, I. M., Sunyaev, R. A.: 1983, *Astrophys. Space Phys. Rev.*, **2**, 189 (Translation).