Second-order gauge-invariant cosmological perturbation theory

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Due to the precise measurement of the Cosmic Microwave Background (CMB) by WMAP, the "precision cosmology" has begun. The farthest region which we can observe by electromagnetic waves is the "last scattering surface" which corresponds to the epoch of the recombination of the hydrogen atom. This last scattering surface is observed as the CMB of 2.7 K by us. CMB is isotropic at the zeroth-order approximation, and its fluctuations (the first-order approximation) have the Gaussian property. Further, just now, Planck is trying to observe non- Gaussian nature (the second-order approximation).

Theory of cosmology should also be developed more precisely. Following to the above observations, it is necessary to construct the theory order by order. The second-order approximation is established by the general-relativistic second-order cosmological perturbation theory. Although the some issues of this theory was already completed in 2003[1], we have made great progress by the recent works.

In general relativity, gravity is represented by curved spacetimes and their metric g_{ab}. In its perturbation theories, we introduce two spacetime manifolds. One is the "physical spacetime" ($\mathcal{M}$, $\bar{g}_{ab}$). We want to clarify the properties of this manifold through perturbations. Another is the "background spacetime" ($\mathcal{M}_0$, g_{ab}) which have nothing to do with $\mathcal{M}$ but we introduce this as a reference of perturbations. We also write the relation between $\bar{g}_{ab}$ and g_{ab} by the perturbative expansion as

$$\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3). \qquad (1)$$

Although $\mathcal{M}$ and $\mathcal{M}_0$ are distinct, through Eq. (1), we implicitly introduce the correspondence of points on these two manifold. This correspondence is called "gauge". However, this correspondence is not unique but there is the degree of freedom in the choice of this correspondence due to the general covariance in the theory. This is called the "gauge degree of freedom". This gauge degree of freedom arises from the relation between $\mathcal{M}$ and $\mathcal{M}_0$ and have nothing to do with the properties of $\mathcal{M}$ itself. Therefore, this gauge degree of freedom is unphysical. In general-relativistic perturbation theory, we have to exclude this gauge degree of freedom to obtain physical results. One of ways to do so is "gauge-invariant perturbation theories". In these theories, we only treat "gauge-invariant perturbative variables" which are independent of the gauge degree of freedom. In linear theories, gauge-invariant formulations have been well-studied, however, in higher-order perturbation theories, it was non-trivial.

In Ref. [1], we assumed that the linear metric perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}, \qquad (2)$$

where $\mathcal{H}_{ab}$ and X_a are gauge-invariant and gauge-dependent parts of h_{ab}, respectively. Based on this assumption, we showed that the second-order metric perturbation l_{ab}, the perturbation Q_1 of first order, and the perturbation Q_2 of the second order for an arbitrary tensor field Q are decomposed as

$$\begin{aligned} l_{ab} &= \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Y - \pounds_X^2\right) g_{ab}, && (3)\\ {}^{(1)}Q &= {}^{(1)}\mathcal{Q} + \pounds_X Q_0, && (4)\\ {}^{(2)}Q &= {}^{(2)}\mathcal{Q} + 2\pounds_X {}^{(1)}Q + \left\{\pounds_Y - \pounds_X^2\right\} Q_0. && (5) \end{aligned}$$

Here, $\mathcal{L}_{ab}$, ${}^{(1)}\mathcal{Q}$, and ${}^{(2)}\mathcal{Q}$ are gauge-invariant part of l_{ab}, ${}^{(1)}Q$, and ${}^{(2)}Q$, respectively. We also showed that perturbations of the spacetime curvatures[2] and perturbations for matter fields[3] are also decomposed as Eqs. (4) and (5). Therefore, these formulae are universal.

In the above formulation, we assumed the decomposition (2) only. Then, we confirmed this assumption in the case of homogeneous isotropic background universe, derived the second-order field equations, and confirmed the consistency of these field equations[4,5.6]. Thus, we constructed the second-order gauge-invariant cosmological perturbation theory, successfully, and it is published as an invited review paper[7]. Finally, we can expect the further development of this second-order gauge-invariant perturbation thoery and comparison with observations.

References

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